3

Static Electric Fields

3–1 Introduction

In Section 1–2 we mentioned that three essential steps are involved in constructing a deductive theory for the study of a scientific subject. They are: the definition of basic quantities, the development of rules of operation, and the postulation of fundamental relations. We have defined the source and field quantities for the electromagnetic model in Chapter 1 and developed the fundamentals of vector algebra and vector calculus in Chapter 2. We are now ready to introduce the fundamental postulates for the study of source-field relationships in electrostatics.

A ***field*** is a spatial distribution of a scalar or vector quantity, which may or may not be a function of time. An example of a scalar is the altitude of a location on a mountain relative to the sea level. It is a scalar, which is not a function of time if long-term erosion and earthquake effects are neglected. Various locations on the mountain have different altitudes, constituting an altitude field. The gradient of altitude is a vector that gives both the direction and the magnitude of the maximum rate of increase (the upward slope) of altitude. On a flat mountaintop or flat land the altitude is constant, and its gradient vanishes. The gravitational field of the earth, representing the force of gravity on a unit mass, is a vector field directed toward the center of the earth, having a magnitude depending on the altitude of the mass. Electric and magnetic field intensities are vector fields.

In electrostatics, electric charges (the sources) are at rest, and electric fields do not change with time. There are no magnetic fields; hence we deal with a relatively simple situation. After we have studied the behavior of static electric fields and mastered the techniques for solving electrostatic boundary-value problems, we will go on to the subject of magnetic fields and time-varying electromagnetic fields. Although electrostatics is relatively simple in the electromagnetics scheme of things, its mastery is fundamental to the understanding of more complicated electromagnetic models. Moreover, the explanation of many natural phenomena (such as lightning, corona, St. Elmo's fire, and grain explosion) and the principles of some important industrial applications (such as oscilloscope, ink-jet printer, xerography, and electret microphone) are based on electrostatics. Many articles on special applications of electrostatics appear in the literature, and a number of books on this subject have also been published.†

The development of electrostatics in elementary physics usually begins with the experimental Coulomb's law (formulated in 1785) for the force between two point charges. This law states that the force between two charged bodies, q_1 and q_2, that are very small in comparison with the distance of separation, R_{12}, is proportional to the product of the charges and inversely proportional to the square of the distance, the direction of the force being along the line connecting the charges. In addition, Coulomb found that unlike charges attract and like charges repel each other. Using vector notation, ***Coulomb's law*** can be written mathematically as

$$\mathbf{F}_{12} = \mathbf{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2}, \tag{3-1}$$

where $\mathbf{F}_{12}$ is the vector force exerted by q_1 on q_2, $\mathbf{a}_{R_{12}}$ is a unit vector in the direction from q_1 to q_2, and k is a proportionality constant depending on the medium and the system of units. Note that if q_1 and q_2 are of the same sign (both positive or both negative), $\mathbf{F}_{12}$ is positive (repulsive); and if q_1 and q_2 are of opposite signs, $\mathbf{F}_{12}$ is negative (attractive). Electrostatics can proceed from Coulomb's law to define electric field intensity $\mathbf{E}$, electric scalar potential V, and electric flux density $\mathbf{D}$, and then lead to Gauss's law and other relations. This approach has been accepted as "logical," perhaps because it begins with an experimental law observed in a laboratory and not with some abstract postulates.

We maintain, however, that Coulomb's law, though based on experimental evidence, is in fact also a postulate. Consider the two stipulations of Coulomb's law: that the charged bodies be very small in comparison with the distance of separation and that the force be inversely proportional to the square of the distance. The question arises regarding the first stipulation: How small must the charged bodies be in order to be considered "very small" in comparison with the distance? In practice the charged bodies cannot be of vanishing sizes (ideal point charges), and there is difficulty in determining the "true" distance between two bodies of finite dimensions. For given body sizes, the relative accuracy in distance measurements is better when the separation is larger. However, practical considerations (weakness of force, existence of extraneous charged bodies, etc.) restrict the usable distance of separation in the laboratory, and experimental inaccuracies cannot be entirely avoided. This leads to a more important question concerning the inverse-square relation of the second

† A. Klinkenberg and J. L. van der Minne, *Electrostatics in the Petroleum Industry*, Elsevier, Amsterdam, 1958. J. H. Dessauer and H. E. Clark, *Xerography and Related Processes*, Focal Press, London, 1965. A. D. Moore (Ed.), *Electrostatics and Its Applications*, John Wiley, New York, 1973. C. E. Jewett, *Electrostatics in the Electronics Environment*, John Wiley, New York, 1976. J.C. Crowley, *Fundamentals of Applied Electrostatics*, John Wiley, New York, 1986.

stipulation. Even if the charged bodies are of vanishing sizes, experimental measurements cannot be of infinite accuracy, no matter how skillful and careful an experimentor is. How then was it possible for Coulomb to know that the force was *exactly* inversely proportional to the *square* (not the 2.000001th or the 1.999999th power) of the distance of separation? This question cannot be answered from an experimental viewpoint because it is not likely that experiments could have been accurate to the seventh place during Coulomb's time.† We must therefore conclude that Coulomb's law is itself a postulate and that the exact relation stipulated by Eq. (3–1) is a law of nature discovered and assumed by Coulomb on the basis of his experiments of limited accuracy.

Instead of following the historical development of electrostatics, we introduce the subject by postulating both the divergence and the curl of the electric field intensity in free space. From Helmholtz's theorem in Section 2–12 we know that a vector field is determined if its divergence and curl are specified. We derive Gauss's law and Coulomb's law from the divergence and curl relations, and we do not present them as separate postulates. The concept of scalar potential follows naturally from a vector identity. Field behaviors in material media will be studied and expressions for electrostatic energy and forces will be developed.

3–2 Fundamental Postulates of Electrostatics in Free Space

We start the study of electromagnetism with the consideration of electric fields due to stationary (static) electric charges in free space. Electrostatics in free space is the simplest special case of electromagnetics. We need to consider only one of the four fundamental vector field quantities of the electromagnetic model discussed in Section 1–2, namely, the electric field intensity **E**. Furthermore, only the permittivity of free space ϵ_0, of the three universal constants mentioned in Section 1–3 enters into our formulation.

Electric field intensity is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists. That is,

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \qquad \text{(V/m).}} \tag{3–2}$$

The electric field intensity **E** is, then, proportional to and in the direction of the force **F**. If **F** is measured in newtons (N) and charge q in coulombs (C), then **E** is in newtons per coulomb (N/C), which is the same as volts per meter (V/m). The test charge

† The exponent on the distance in Coulomb's law has been verified by an indirect experiment to be 2 to within one part in 10^{15}. (See E. R. Williams, J. E. Faller, and H. A. Hall, *Phys. Rev. Letters*, vol. 26, 1971, p. 721.)

q, of course, cannot be zero in practice; as a matter of fact, it cannot be less than the charge on an electron. However, the finiteness of the test charge would not make the measured $\mathbf{E}$ differ appreciably from its calculated value if the test charge is small enough not to disturb the charge distribution of the source. An inverse relation of Eq. (3–2) gives the force $\mathbf{F}$ on a stationary charge q in an electric field $\mathbf{E}$:

$$\boxed{\mathbf{F} = q\mathbf{E} \qquad \text{(N)}.} \tag{3-3}$$

The two fundamental postulates of electrostatics in free space specify the divergence and curl of $\mathbf{E}$. They are

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{3-4}$$

and

$$\boxed{\nabla \times \mathbf{E} = 0.} \tag{3-5}$$

In Eq. (3–4), ρ is the volume charge density of free charges (C/m^3), and ϵ_0 is the permittivity of free space, a universal constant.† Equation (3–5) asserts that ***static electric fields are irrotational***, whereas Eq. (3–4) implies that a static electric field is not solenoidal unless $\rho = 0$. These two postulates are concise, simple, and independent of any coordinate system; and they can be used to derive all other relations, laws, and theorems in electrostatics! Such is the beauty of the deductive, axiomatic approach.

Equations (3–4) and (3–5) are point relations; that is, they hold at every point in space. They are referred to as the differential form of the postulates of electrostatics, since both divergence and curl operations involve spatial derivatives. In practical applications we are usually interested in the total field of an aggregate or a distribution of charges. This is more conveniently obtained by an integral form of Eq. (3–4). Taking the volume integral of both sides of Eq. (3–4) over an arbitrary volume V, we have

$$\int_V \nabla \cdot \mathbf{E}\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv. \tag{3-6}$$

In view of the divergence theorem in Eq. (2–115), Eq. (3–6) becomes

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},} \tag{3-7}$$

† The permittivity of free space $\epsilon_0 \cong \frac{1}{36\pi} \times 10^{-9}$ (F/m). See Eq. (1–11).

where Q is the total charge contained in volume V bounded by surface S. Equation (3–7) is a form of ***Gauss's law***, which states that ***the total outward flux of the electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface divided by*** ϵ_0. Gauss's law is one of the most important relations in electrostatics. We will discuss it further in Section 3–4, along with illustrative examples.

An integral form can also be obtained for the curl relation in Eq. (3–5) by integrating $\nabla \times \mathbf{E}$ over an open surface and invoking Stokes's theorem as expressed in Eq. (2–143). We have

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.} \tag{3–8}$$

The line integral is performed over a closed contour C bounding an arbitrary surface; hence C is itself arbitrary. As a matter of fact, the surface does not even enter into Eq. (3–8), which asserts that ***the scalar line integral of the static electric field intensity around any closed path vanishes.*** The scalar product $\mathbf{E} \cdot d\boldsymbol{\ell}$ integrated over any path is the voltage along that path. Thus Eq. (3–8) is an expression of ***Kirchhoff's voltage law*** in circuit theory that ***the algebraic sum of voltage drops around any closed circuit is zero.*** This will be discussed again in Section 5–3.

Equation (3–8) is another way of saying that $\mathbf{E}$ is irrotational (conservative). Referring to Fig. 3–1, we see that if the scalar line integral of $\mathbf{E}$ over the arbitrary closed contour C_1C_2 is zero, then

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} + \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \tag{3–9}$$

or

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = - \underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E} \cdot d\boldsymbol{\ell} \tag{3–10}$$

or

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell}. \tag{3–11}$$

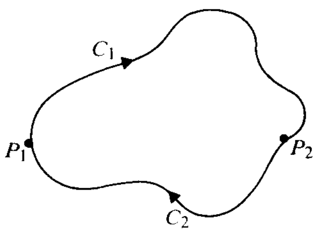


FIGURE 3–1
An arbitrary contour.

Equation (3–11) says that the scalar line integral of the irrotational **E** field is independent of the path; it depends only on the end points. As we shall see in Section 3–5, the integrals in Eq. (3–11) represent the work done *by* the electric field in moving a unit charge from point P_1 to point P_2; hence Eqs. (3–8) and (3–9) imply a statement of conservation of work or energy in an electrostatic field.

The two fundamental postulates of electrostatics in free space are repeated below because they form the foundation upon which we build the structure of electrostatics.

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

We consider these postulates, like the principle of conservation of charge, to be representations of laws of nature. In the following section we shall *derive* Coulomb's law.

3–3 Coulomb's Law

We consider the simplest possible electrostatic problem of a single point charge, q, at rest in a boundless free space. In order to find the electric field intensity due to q, we draw a hypothetical spherical surface of a radius R centered at q. Since a point charge has no preferred directions, its electric field must be everywhere radial and has the same intensity at all points on the spherical surface. Applying Eq. (3–7) to Fig. 3–2(a), we have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

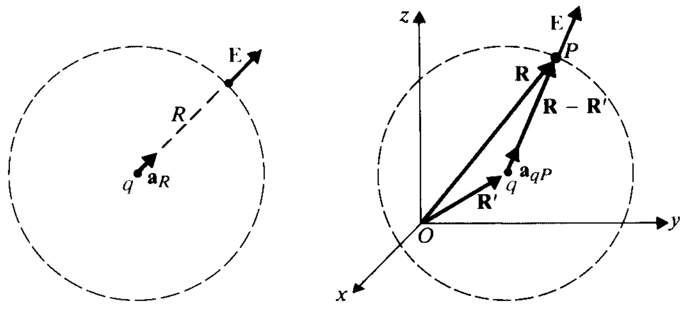
or

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$

Therefore,

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \qquad (\text{V/m}).} \tag{3–12}$$

Equation (3–12) tells us that ***the electric field intensity of a positive point charge is in the outward radial direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.*** This is a very important basic formula in electrostatics. Using Eq. (2–139), we can verify that



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGURE due to a point charge.

$\nabla \times \mathbf{E} = 0$ for the $\mathbf{E}$ given in Eq. (3–12). A flux-line graph for the electric field intensity of a positive point charge q will look like Fig. 2–25(b).

If the charge q is not located at the origin of a chosen coordinate system, suitable changes should be made to the unit vector $\mathbf{a}_R$ and the distance R to reflect the locations of the charge and of the point at which $\mathbf{E}$ is to be determined. Let the position vector of q be $\mathbf{R}'$ and that of a field point P be $\mathbf{R}$, as shown in Fig. 3–2(b). Then, from Eq. (3–12),

$$\mathbf{E}_P = \mathbf{a}_{qP} \frac{q}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^2}, \tag{3–13}$$

where $\mathbf{a}_{qP}$ is the unit vector drawn from q to P. Since

$$\mathbf{a}_{qP} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|} \tag{3–14}$$

we have

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad \text{(V/m).}} \tag{3–15}$$

EXAMPLE 3–1 Determine the electric field intensity at $P(-0.2, 0, -2.3)$ due to a point charge of $+5$ (nC) at $Q(0.2, 0.1, -2.5)$ in air. All dimensions are in meters.

Solution The position vector for the field point P

$$\mathbf{R} = \overrightarrow{OP} = -\mathbf{a}_x 0.2 - \mathbf{a}_z 2.3.$$

The position vector for the point charge Q is

$$\mathbf{R}' = \overrightarrow{OQ} = \mathbf{a}_x 0.2 + \mathbf{a}_y 0.1 - \mathbf{a}_z 2.5.$$

The difference is

$$\mathbf{R} - \mathbf{R}' = -\mathbf{a}_x 0.4 - \mathbf{a}_y 0.1 + \mathbf{a}_z 0.2,$$

which has a magnitude

$$|\mathbf{R} - \mathbf{R}'| = [(-0.4)^2 + (-0.1)^2 + (0.2)^2]^{1/2} = 0.458 \text{ (m)}.$$

Substituting in Eq. (3–15), we obtain

$$\begin{aligned}\mathbf{E}_P &= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3} \\ &= (9 \times 10^9)\frac{5 \times 10^{-9}}{0.458^3}(-\mathbf{a}_x 0.4 - \mathbf{a}_y 0.1 + \mathbf{a}_z 0.2) \\ &= 214.5(-\mathbf{a}_x 0.873 - \mathbf{a}_y 0.218 + \mathbf{a}_z 0.437) \quad \text{(V/m)}.\end{aligned}$$

The quantity within the parentheses is the unit vector $\mathbf{a}_{QP} = (\mathbf{R} - \mathbf{R}')/|\mathbf{R} - \mathbf{R}'|$, and $\mathbf{E}_P$ has a magnitude of 214.5 (V/m). ■

Note: The permittivity of air is essentially the same as that of the free space. The factor $1/(4\pi\epsilon_0)$ appears very frequently in electrostatics. From Eq. (1–11) we know that $\epsilon_0 = 1/(c^2\mu_0)$. But $\mu_0 = 4\pi \times 10^{-7}$ (H/m) in SI units; so

$$\frac{1}{4\pi\epsilon_0} = \frac{\mu_0 c^2}{4\pi} = 10^{-7}\, c^2 \qquad \text{(m/F)} \tag{3–16}$$

exactly. If we use the approximate value $c = 3 \times 10^8$ (m/s), then $1/(4\pi\epsilon_0) = 9 \times 10^9$ (m/F).

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force $\mathbf{F}_{12}$ is experienced by q_2 due to electric field intensity $\mathbf{E}_{12}$ of q_1 at q_2. Combining Eqs. (3–3) and (3–12), we have

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \qquad \text{(N)}.} \tag{3–17}$$

Equation (3–17) is a mathematical form of ***Coulomb's law*** already stated in Section 3–1 in conjunction with Eq. (3–1). Note that the exponent on R is *exactly* 2, which is a consequence of the fundamental postulate Eq. (3–4). In SI units the proportionality constant k equals $1/(4\pi\epsilon_0)$, and the force is in newtons (N).

EXAMPLE 3–2 A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.

Solution We shall solve this problem in two ways.

a) At any point, such as P, inside the hollow shell shown in Fig. 3–3, an arbitrary hypothetical closed surface (a ***Gaussian surface***) may be drawn, over which we apply Gauss's law, Eq. (3–7). Since no charge exists inside the shell and the surface is arbitrary, we conclude readily that $\mathbf{E} = 0$ everywhere inside the shell.

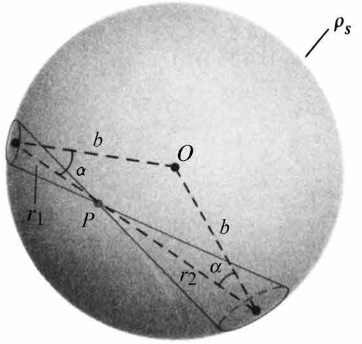


FIGURE 3–3
A charged shell (Example 3–2).

b) Let us now examine the problem in more detail. Draw a pair of elementary cones of solid angle $d\Omega$ with vertex at an arbitrary point P. The cones extend in both directions, intersecting the shell in areas ds_1 and ds_2 at distances r_1 and r_2, respectively, from the point P. Since charge Q distributes uniformly over the spherical shell, there is a uniform surface charge density

$$\rho_s = \frac{Q}{4\pi b^2}. \tag{3–18}$$

The magnitude of the electric field intensity at P due to charges on the elementary surfaces ds_1 and ds_2 is, from Eq. (3–12),

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{ds_1}{r_1^2} - \frac{ds_2}{r_2^2}\right). \tag{3–19}$$

But the solid angle $d\Omega$ equals

$$d\Omega = \frac{ds_1}{r_1^2}\cos\alpha = \frac{ds_2}{r_2^2}\cos\alpha. \tag{3–20}$$

Combining the expressions of dE and $d\Omega$, we find that

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha}\right) = 0. \tag{3–21}$$

Since the above result applies to every pair of elementary cones, we conclude that $\mathbf{E} = 0$ everywhere inside the conducting shell, as before. ■

It will be noted that if Coulomb's law as expressed in Eq. (3–12) and used in Eq. (3–19) was slightly different from an inverse-square relation, the substitution of Eq. (3–20), which is a geometrical relation, in Eq. (3–19) would not yield the result $dE = 0$. Consequently, the electric field intensity inside the shell would not vanish; indeed, it would vary with the location of the point P. Coulomb originally used a torsion balance to conduct his experiments, which were necessarily of limited accuracy. Nevertheless, he was brilliant enough to *postulate* the inverse-square law. Many scientists subsequently made use of the vanishing field inside a spherical shell illustrated in this example to verify the inverse-square law. The field inside a charged shell, if it existed, could be detected to a very high accuracy by a probe through a small hole in the shell.

EXAMPLE 3–3 The electrostatic deflection system of a cathode-ray oscilloscope is depicted in Fig. 3–4. Electrons from a heated cathode are given an initial velocity $\mathbf{u}_0 = \mathbf{a}_z u_0$ by a positively charged anode (not shown). The electrons enter at $z = 0$ into a region of deflection plates where a uniform electric field $\mathbf{E}_d = -\mathbf{a}_y E_d$ is maintained over a width w. Ignoring gravitational effects, find the vertical deflection of the electrons on the fluorescent screen at $z = L$.

Solution Since there is no force in the z-direction in the $z > 0$ region, the horizontal velocity u_0 is maintained. The field $\mathbf{E}_d$ exerts a force on the electrons each carrying a charge $-e$, causing a deflection in the y-direction:

$$\mathbf{F} = (-e)\mathbf{E}_d = \mathbf{a}_y e E_d.$$

From Newton's second law of motion in the vertical direction we have

$$m \frac{du_y}{dt} = eE_d,$$

where m is the mass of an electron. Integrating both sides, we obtain

$$u_y = \frac{dy}{dt} = \frac{e}{m} E_d t,$$

where the constant of integration is set to zero because $u_y = 0$ at $t = 0$. Integrating again, we have

$$y = \frac{e}{2m} E_d t^2.$$

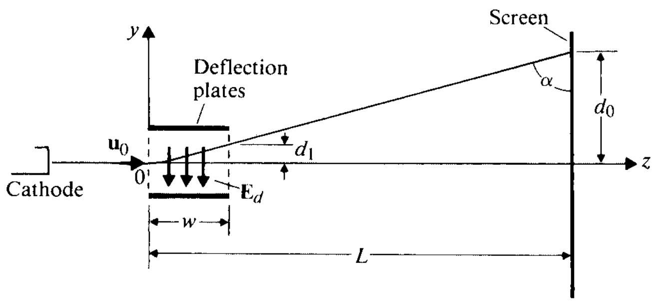


FIGURE 3–4
Electrostatic deflection system of a cathode-ray oscilloscope (Example 3–3)

The constant of integration is again zero because $y = 0$ at $t = 0$. Note that the electrons have a parabolic trajectory between the deflection plates. At the exit from the deflection plates, $t = w/u_0$,

$$d_1 = \frac{eE_d}{2m}\left(\frac{w}{u_0}\right)^2$$

and

$$u_{y1} = u_y\left(t = \frac{w}{u_0}\right) = \frac{eE_d}{m}\left(\frac{w}{u_0}\right).$$

When the electrons reach the screen, they have traveled a further horizontal distance of $(L - w)$ which takes $(L - w)/u_0$ seconds. During that time there is an additional vertical deflection

$$d_2 = u_{y1}\left(\frac{L - w}{u_0}\right) = \frac{eE_d}{m}\frac{w(L - w)}{u_0^2}.$$

Hence the deflection at the screen is

$$d_0 = d_1 + d_2 = \frac{eE_d}{mu_0^2}w\left(L - \frac{w}{2}\right).$$

Ink-jet printers used in computer output, like cathode-ray oscilloscopes, are devices based on the principle of electrostatic deflection of a stream of charged particles. Minute droplets of ink are forced through a vibrating nozzle controlled by a piezoelectric transducer. The output of the computer imparts variable amounts of charges on the ink droplets, which then pass through a pair of deflection plates where a uniform static electric field exists. The amount of droplet deflection depends on the charge it carries, causing the ink jet to strike the print surface and form an image as the print head moves in a horizontal direction.

3–3.1 ELECTRIC FIELD DUE TO A SYSTEM OF DISCRETE CHARGES

Suppose an electrostatic field is created by a group of n discrete point charges q_1, $q_2, \ldots, q_n$ located at different positions. Since electric field intensity is a linear function of (proportional to) $\mathbf{a}_R q/R^2$, the principle of superposition applies, and the total **E** field at a point is the *vector sum* of the fields caused by all the individual charges. From Eq. (3–15) we can write the electric intensity at a field point whose position vector is **R** as

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \quad \text{(V/m)}.} \qquad (3\text{–}22)$$

Although Eq. (3–22) is a succinct expression, it is somewhat inconvenient to use because of the need to add vectors of different magnitudes and directions.

Let us consider the simple case of an ***electric dipole*** that consists of a pair of equal and opposite charges $+q$ and $-q$, separated by a small distance, d, as shown in Fig. 3–5. Let the center of the dipole coincide with the origin of a spherical coordinate system. Then the **E** field at the point P is the sum of the contributions due to $+q$ and $-q$. Thus,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^3}\right\}. \tag{3-23}$$

The first term on the right side of Eq. (3–23) can be simplified if $d \ll R$. We write

$$\begin{aligned}\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^{-3} &= \left[\left(\mathbf{R} - \frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R} - \frac{\mathbf{d}}{2}\right)\right]^{-3/2} \\ &= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2} \\ &\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \\ &\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],\end{aligned} \tag{3-24}$$

where the binomial expansion has been used and all terms containing the second and higher powers of (d/R) have been neglected. Similarly, for the second term on the right side of Eq. (3–23) we have

$$\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]. \tag{3-25}$$

Substitution of Eqs. (3–24) and (3–25) in Eq. (3–23) leads to

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]. \tag{3-26}$$

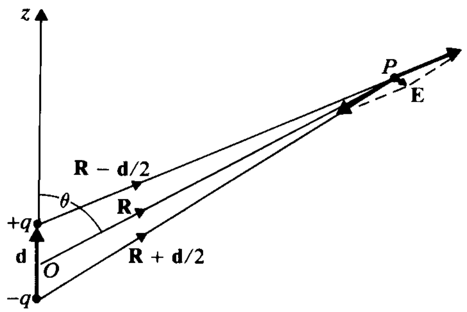


FIGURE 3–5
Electric field of a dipole.

The derivation and interpretation of Eq. (3–26) require the manipulation of vector quantities. We can appreciate that determining the electric field caused by three or more discrete charges will be even more tedious. In Section 3–5 we will introduce the concept of a scalar electric potential, with which the electric field intensity caused by a distribution of charges can be found more easily.

The electric dipole is an important entity in the study of the electric field in dielectric media. We define the product of the charge q and the vector $\mathbf{d}$ (going from $-q$ to $+q$) as the ***electric dipole moment***, $\mathbf{p}$:

$$\mathbf{p} = q\mathbf{d}. \tag{3–27}$$

Equation (3–26) can then be rewritten as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right], \tag{3–28}$$

where the approximate sign ($\sim$) over the equal sign has been left out for simplicity. If the dipole lies along the z-axis as in Fig. 3–5, then (see Eq. 2–77)

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta), \tag{3–29}$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta, \tag{3–30}$$

and Eq. (3–28) becomes

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(V/m)}.} \tag{3–31}$$

Equation (3–31) gives the electric field intensity of an electric dipole in spherical coordinates. We see that $\mathbf{E}$ of a dipole is inversely proportional to the cube of the distance R. This is reasonable because as R increases, the fields due to the closely spaced $+q$ and $-q$ tend to cancel each other more completely, thus decreasing more rapidly than that of a single point charge.

3–3.2 ELECTRIC FIELD DUE TO A CONTINUOUS DISTRIBUTION OF CHARGE

The electric field caused by a continuous distribution of charge can be obtained by integrating (superposing) the contribution of an element of charge over the charge distribution. Refer to Fig. 3–6, where a volume charge distribution is shown. The volume charge density ρ (C/m^3) is a function of the coordinates. Since a differential element of charge behaves like a point charge, the contribution of the charge $\rho\,dv'$ in a differential volume element dv' to the electric field intensity at the field point P is

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\,dv'}{4\pi\epsilon_0 R^2}. \tag{3–32}$$

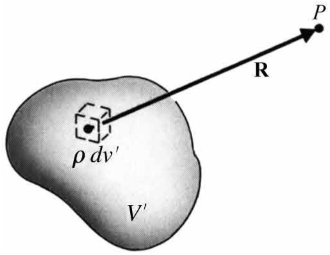


FIGURE 3–6
Electric field due to a continuous charge distribution.

We have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)}, \tag{3-33}$$

or, since $\mathbf{a}_R = \mathbf{R}/R$,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}. \tag{3-34}$$

Except for some especially simple cases, the vector triple integral in Eq. (3–33) or Eq. (3–34) is difficult to carry out because, in general, all three quantities in the integrand ($\mathbf{a}_R$, ρ, and R) change with the location of the differential volume dv'.

If the charge is distributed on a surface with a surface charge density ρ_s (C/m^2), then the integration is to be carried out over the surface (not necessarily flat). Thus,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m)}. \tag{3-35}$$

For a line charge we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m)}, \tag{3-36}$$

where ρ_ℓ (C/m) is the line charge density, and L' the line (not necessarily straight) along which the charge is distributed.

EXAMPLE 3–4 Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

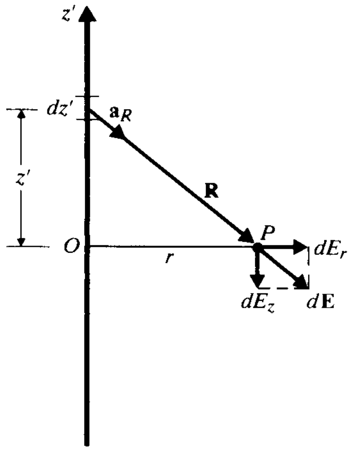


FIGURE 3–7
An infinitely long, straight, line charge.

Solution Let us assume that the line charge lies along the z'-axis as shown in Fig. 3–7. (We are perfectly free to do this because the field obviously does not depend on how we designate the line. *It is an accepted convention to use primed coordinates for source points and unprimed coordinates for field points when there is a possibility of confusion.*) The problem asks us to find the electric field intensity at a point P, which is at a distance r from the line. Since the problem has a cylindrical symmetry (that is, the electric field is independent of the azimuth angle ϕ), it would be most convenient to work with cylindrical coordinates. We rewrite Eq. (3–36) as

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \rho_\ell \frac{\mathbf{R}}{R^3}\, d\ell' \qquad \text{(V/m).}} \tag{3–37}$$

For the problem at hand, ρ_ℓ is constant, and a line element $d\ell' = dz'$ is chosen to be at an arbitrary distance z' from the origin. It is most important to remember that $\mathbf{R}$ is the distance vector directed *from the source to the field point*, not the other way around. We have

$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'. \tag{3–38}$$

The electric field, $d\mathbf{E}$, due to the differential line charge element $\rho_\ell\, d\ell' = \rho_\ell\, dz'$ is

$$\begin{aligned} d\mathbf{E} &= \frac{\rho_\ell\, dz'}{4\pi\epsilon_0} \frac{\mathbf{a}_r r - \mathbf{a}_z z'}{(r^2 + z'^2)^{3/2}} \\ &= \mathbf{a}_r\, dE_r + \mathbf{a}_z\, dE_z, \end{aligned} \tag{3–39}$$

where

$$dE_r = \frac{\rho_\ell r\, dz'}{4\pi\epsilon_0 (r^2 + z'^2)^{3/2}} \tag{3–39a}$$

and

$$dE_z = \frac{-\rho_\ell z'\, dz'}{4\pi\epsilon_0(r^2 + z'^2)^{3/2}}. \tag{3–39b}$$

In Eq. (3–39) we have decomposed $d\mathbf{E}$ into its components in the $\mathbf{a}_r$ and $\mathbf{a}_z$ directions. It is easy to see that for every $\rho_\ell\, dz'$ at $+z'$ there is a charge element $\rho_\ell\, dz'$ at $-z'$, which will produce a $d\mathbf{E}$ with components dE_r and $-dE_z$. Hence the $\mathbf{a}_z$ components will cancel in the integration process, and we only need to integrate the dE_r in Eq. (3–39a):

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{\rho_\ell r}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{dz'}{(r^2 + z'^2)^{3/2}}$$

or

$$\boxed{\mathbf{E} = \mathbf{a}_r \frac{\rho_\ell}{2\pi\epsilon_0 r} \qquad (\text{V/m}).} \tag{3–40}$$

■

Equation (3–40) is an important result for an infinite line charge. Of course, no physical line charge is infinitely long; nevertheless, Eq. (3–40) gives the approximate **E** field of a long straight line charge at a point close to the line charge.

3–4 Gauss's Law and Applications

Gauss's law follows directly from the divergence postulate of electrostatics, Eq. (3–4), by the application of the divergence theorem. It was derived in Section 3–2 as Eq. (3–7) and is repeated here on account of its importance:

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.} \tag{3–41}$$

Gauss's law asserts that the total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ϵ_0. We note that the surface S can be *any hypothetical (mathematical) closed surface chosen for convenience*; it does not have to be, and usually is not, a physical surface.

Gauss's law is particularly useful in determining the E-field of charge distributions with some symmetry conditions, such that *the normal component of the electric field intensity is constant over an enclosed surface.* In such cases the surface integral on the left side of Eq. (3–41) would be very easy to evaluate, and Gauss's law would be a much more efficient way for finding the electric field intensity than Eqs. (3–33) through (3–37). On the other hand, when symmetry conditions do not exist, Gauss's law would not be of much help. The essence of applying Gauss's law lies first in the recognition of symmetry conditions and second in the suitable choice of a surface over which the normal component of **E** resulting from a given charge distribution is a constant. Such a surface is referred to as a ***Gaussian surface***. This basic principle was used to obtain Eq. (3–12) for a point charge that possesses spherical symmetry; consequently, a proper Gaussian surface is the surface of a sphere centered at the point charge. Gauss's law could not help in the derivation of Eq. (3–26) or Eq. (3–31) for an electric dipole, since a surface about a separated pair of equal and opposite charges over which the normal component of **E** remains constant was not known.

EXAMPLE 3–5 Use Gauss's law to determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

Solution This problem was solved in Example 3–4 by using Eq. (3–36). Since the line charge is infinitely long, the resultant **E** field must be radial and perpendicular to the line charge ($\mathbf{E} = \mathbf{a}_r E_r$), and a component of **E** along the line cannot exist. With the obvious cylindrical symmetry we construct a cylindrical Gaussian surface of a radius r and an arbitrary length L with the line charge as its axis, as shown in Fig. 3–8. On this surface, E_r is constant, and $d\mathbf{s} = \mathbf{a}_r r\, d\phi\, dz$ (from Eq. 2–53a). We have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \int_0^L \int_0^{2\pi} E_r r\, d\phi\, dz = 2\pi r L E_r.$$

There is no contribution from the top or the bottom face of the cylinder because on the top face $d\mathbf{s} = \mathbf{a}_z r\, dr\, d\phi$ but **E** has no z-component there, making $\mathbf{E} \cdot d\mathbf{s} = 0$. Similarly for the bottom face. The total charge enclosed in the cylinder is $Q = \rho_\ell L$. Substitution into Eq. (3–41) gives us immediately

$$2\pi r L E_r = \frac{\rho_\ell L}{\epsilon_0}$$

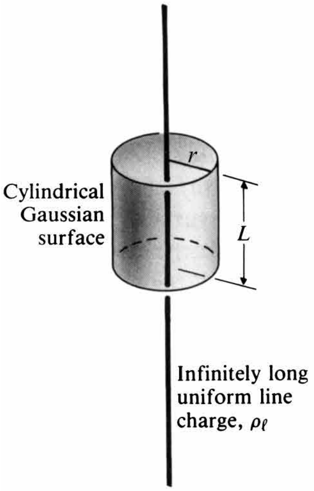


FIGURE 3–8
Applying Gauss's law to an infinitely long line charge (Example 3–5).

or

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{\rho_\ell}{2\pi\epsilon_0 r}.$$

This result is, of course, the same as that given in Eq. (3–40), but it is obtained here in a much simpler way. We note that the length L of the cylindrical Gaussian surface does not appear in the final expression; hence we could have chosen a cylinder of a unit length.

EXAMPLE 3–6 Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.

Solution It is clear that the **E** field caused by a charged sheet of an infinite extent is normal to the sheet. Equation (3–35) could be used to find **E**, but this would involve a double integration between infinite limits of a general expression of $1/R^2$. Gauss's law can be used to much advantage here.

We choose as the Gaussian surface a rectangular box with top and bottom faces of an arbitrary area A equidistant from the planar charge, as shown in Fig. 3–9. The sides of the box are perpendicular to the charged sheet. If the charged sheet coincides with the xy-plane, then on the top face,

$$\mathbf{E} \cdot d\mathbf{s} = (\mathbf{a}_z E_z) \cdot (\mathbf{a}_z\, ds) = E_z\, ds.$$

On the bottom face,

$$\mathbf{E} \cdot d\mathbf{s} = (-\mathbf{a}_z E_z) \cdot (-\mathbf{a}_z\, ds) = E_z\, ds.$$

Since there is no contribution from the side faces, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = 2E_z \int_A ds = 2E_z A.$$

The total charge enclosed in the box is $Q = \rho_s A$. Therefore,

$$2E_z A = \frac{\rho_s A}{\epsilon_0},$$

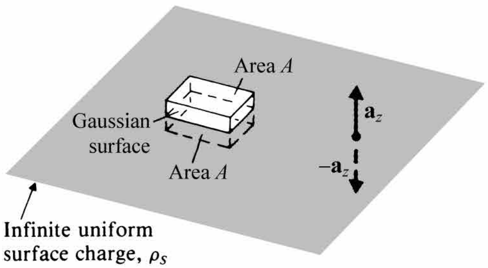


FIGURE 3–9
Applying Gauss's law to an infinite planar charge (Example 3–6).

from which we obtain

$$\boxed{\mathbf{E} = \mathbf{a}_z E_z = \mathbf{a}_z \frac{\rho_s}{2\epsilon_0}, \qquad z > 0,} \tag{3-42a}$$

and

$$\boxed{\mathbf{E} = -\mathbf{a}_z E_z = -\mathbf{a}_z \frac{\rho_s}{2\epsilon_0}, \qquad z < 0.} \tag{3-42b}$$

Of course, the charged sheet may not coincide with the xy-plane (in which case we do not speak in terms of above and below the plane), but the **E** field always points *away* from the sheet if ρ_s is *positive*. It is obvious that the Gaussian surface could have been a pillbox of any shape, not necessarily rectangular. ▬

The lighting scheme of an office or a classroom may consist of incandescent bulbs, long fluorescent tubes, or ceiling panel lights. These correspond roughly to point sources, line sources, and planar sources, respectively. From Eqs. (3–12), (3–40), and (3–42) we can estimate that light intensity will fall off rapidly as the square of the distance from the source in the case of incandescent bulbs, less rapidly as the first power of the distance for long fluorescent tubes, and not at all for ceiling panel lights.

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \le R \le b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

Solution First we recognize that the given source condition has spherical symmetry. The proper Gaussian surfaces must therefore be concentric spherical surfaces. We must find the **E** field in two regions. Refer to Fig. 3–10.

a) $0 \le R \le b$

A hypothetical spherical Gaussian surface S_i with $R < b$ is constructed within the electron cloud. On this surface, **E** is radial and has a constant magnitude:

$$\mathbf{E} = \mathbf{a}_R E_R, \qquad d\mathbf{s} = \mathbf{a}_R\, ds.$$

The total outward E flux is

$$\oint_{S_i} \mathbf{E} \cdot d\mathbf{s} = E_R \int_{S_i} ds = E_R 4\pi R^2.$$

The total charge enclosed within the Gaussian surface is

$$Q = \int_V \rho\, dv$$
$$= -\rho_o \int_V dv = -\rho_o \frac{4\pi}{3} R^3.$$

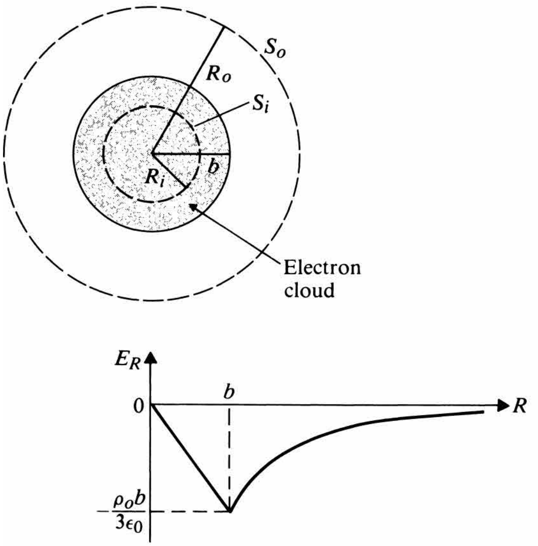


FIGURE 3–10
Electric field intensity of a spherical electron cloud (Example 3–7).

Substitution into Eq. (3–7) yields

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o}{3\epsilon_0} R, \qquad 0 \le R \le b.$$

We see that within the uniform electron cloud the **E** field is directed toward the center and has a magnitude proportional to the distance from the center.

b) $R \ge b$
For this case we construct a spherical Gaussian surface S_o with $R > b$ outside the electron cloud. We obtain the same expression for $\oint_{S_o} \mathbf{E} \cdot d\mathbf{s}$ as in case (a). The total charge enclosed is

$$Q = -\rho_o \frac{4\pi}{3} b^3.$$

Consequently,

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o b^3}{3\epsilon_0 R^2}, \qquad R \ge b,$$

which follows the inverse square law and could have been obtained directly from Eq. (3–12). We observe that *outside* the charged cloud the **E** field is exactly the same as though the total charge is concentrated on a single point charge at the center. This is true, in general, for a spherically symmetrical charged region even though ρ is a function of R. ▬

The variation of E_R versus R is plotted in Fig. 3–10. Note that the formal solution of this problem requires only a few lines. If Gauss's law is not used, it is necessary (1) to choose a differential volume element arbitrarily located in the electron cloud, (2) to express its vector distance **R** to a field point in a chosen coordinate system, and (3) to perform a triple integration as indicated in Eq. (3–33). This is a hopelessly involved process. The moral is: *Try to apply Gauss's law if symmetry conditions exist for the given charge distribution.*

3–5 Electric Potential

In connection with the null identity in Eq. (2–145) we noted that a curl-free vector field could always be expressed as the gradient of a scalar field. This induces us to *define* a scalar ***electric potential*** V such that

$$\boxed{\mathbf{E} = -\nabla V} \tag{3–43}$$

because scalar quantities are easier to handle than vector quantities. If we can determine V more easily, then **E** can be found by a gradient operation, which is a straightforward process in an orthogonal coordinate system. The reason for the inclusion of a negative sign in Eq. (3–43) will be explained presently.

Electric potential does have physical significance, and it is related to the work done in carrying a charge from one point to another. In Section 3–2 we defined the electric field intensity as the force acting on a unit test charge. Therefore in moving a unit charge from point P_1 to point P_2 in an electric field, work must be done *against the field* and is equal to

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V)}. \tag{3–44}$$

Many paths may be followed in going from P_1 to P_2. Two such paths are drawn in Fig. 3–11. Since the path between P_1 and P_2 is not specified in Eq. (3–44), the

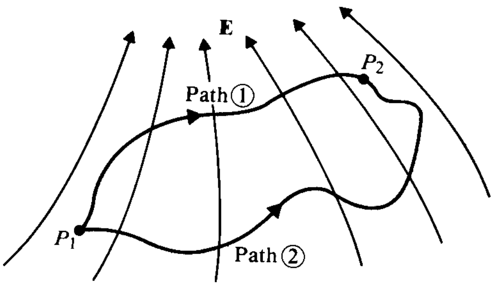


FIGURE 3–11
Two paths leading from P_1 to P_2 in an electric field.

question naturally arises, how does the work depend on the path taken? A little thought will lead us to conclude that W/q in Eq. (3–44) should not depend on the path; if it did, one would be able to go from P_1 to P_2 along a path for which W is smaller and then to come back to P_1 along another path, achieving a net gain in work or energy. This would be contrary to the principle of conservation of energy. We have already alluded to the path-independent nature of the scalar line integral of the irrotational (conservative) **E** field when we discussed Eq. (3–8).

Analogous to the concept of potential energy in mechanics, Eq. (3–44) represents the difference in electric potential energy of a unit charge between point P_2 and point P_1. Denoting the electric potential energy per unit charge by V, the ***electric potential***, we have

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell \qquad \text{(V)}.} \tag{3–45}$$

Mathematically, Eq. (3–45) can be obtained by substituting Eq. (3–43) in Eq. (3–44). Thus, in view of Eq. (2–88),

$$\begin{aligned} -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell &= \int_{P_1}^{P_2} (\nabla V)\cdot(\mathbf{a}_\ell\, d\ell) \\ &= \int_{P_1}^{P_2} dV = V_2 - V_1. \end{aligned}$$

What we have defined in Eq. (3–45) is a ***potential difference*** (***electrostatic voltage***) between points P_2 and P_1. It makes no more sense to talk about the absolute potential of a point than about the absolute phase of a phasor or the absolute altitude of a geographical location; a reference zero-potential point, a reference zero phase (usually at $t = 0$), or a reference zero altitude (usually at sea level) must first be specified. In most (but not all) cases the zero-potential point is taken at infinity. When the reference zero-potential point is not at infinity, it should be specifically stated.

We want to make two more points about Eq. (3–43). First, the inclusion of the negative sign is necessary in order to conform with the convention that in going *against* the **E** field the electric potential V *increases*. For instance, when a d-c battery of a voltage V_0 is connected between two parallel conducting plates, as in Fig. 3–12, positive and negative charges cumulate on the top and bottom plates, respectively. The **E** field is directed from positive to negative charges, while the potential increases in the *opposite* direction. Second, we know from Section 2–6, when we defined the gradient of a scalar field, that the direction of ∇V is normal to the surfaces of constant

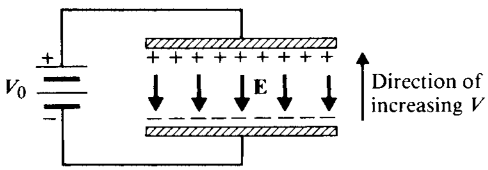


FIGURE 3–12
Relative directions of **E** and increasing V.

V. Hence if we use directed ***field lines*** or ***streamlines*** to indicate the direction of the **E** field, they are everywhere perpendicular to ***equipotential lines*** and ***equipotential surfaces***.

3–5.1 ELECTRIC POTENTIAL DUE TO A CHARGE DISTRIBUTION

The electric potential of a point at a distance R from a point charge q referred to that at infinity can be obtained readily from Eq. (3–45):

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}\right) \cdot (\mathbf{a}_R \, dR), \tag{3–46}$$

which gives

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V).}} \tag{3–47}$$

This is a scalar quantity and depends on, besides q, only the distance R. The potential difference between any two points P_2 and P_1 at distances R_2 and R_1, respectively, from q is

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right). \tag{3–48}$$

This result may appear a little surprising at first, since P_2 and P_1 may not lie on the same radial line through q, as illustrated in Fig. 3–13. However, the concentric circles (spheres) passing through P_2 and P_1 are equipotential lines (surfaces), and $V_{P_2} - V_{P_1}$ is the same as $V_{P_2} - V_{P_3}$. From the point of view of Eq. (3–45) we can choose the path of integration from P_1 to P_3 and then from P_3 to P_2. No work is done from P_1 to P_3 because **F** is perpendicular to $d\boldsymbol{\ell} = \mathbf{a}_\phi R_1 \, d\phi$ along the circular path ($\mathbf{E} \cdot d\boldsymbol{\ell} = 0$).

The electric potential at **R** due to a system of n discrete point charges $q_1, q_2, \ldots, q_n$ located at $\mathbf{R}'_1, \mathbf{R}'_2, \ldots, \mathbf{R}'_n$ is, by superposition, the sum of the potentials due to

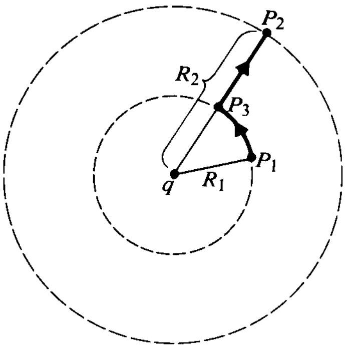


FIGURE 3–13
Path of integration about a point charge.

the individual charges:

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad \text{(V)}. \tag{3–49}$$

Since this is a scalar sum, it is, in general, easier to determine **E** by taking the negative gradient of V than from the vector sum in Eq. (3–22) directly.

As an example, let us again consider an electric dipole consisting of charges $+q$ and $-q$ with a small separation d. The distances from the charges to a field point P are designated R_+ and R_-, as shown in Fig. 3–14. The potential at P can be written down directly:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right). \tag{3–50}$$

If $d \ll R$, we have

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right) \tag{3–51}$$

and

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right). \tag{3–52}$$

Substitution of Eqs. (3–51) and (3–52) in Eq. (3–50) gives

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2} \tag{3–53a}$$

or

$$V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \quad \text{(V)}, \tag{3–53b}$$

where $\mathbf{p} = q\mathbf{d}$. (The "approximate" sign ($\sim$) has been dropped for simplicity.)

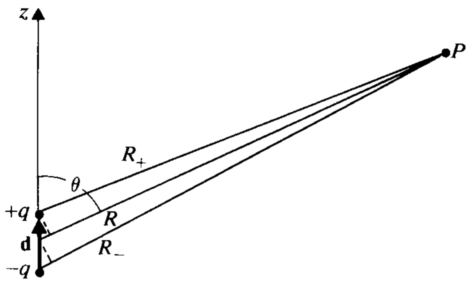


FIGURE 3–14
An electric dipole.

The **E** field can be obtained from $-\nabla V$. In spherical coordinates we have

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\mathbf{a}_R \frac{\partial V}{\partial R} - \mathbf{a}_\theta \frac{\partial V}{R\,\partial \theta} \\ &= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).\end{aligned} \tag{3-54}$$

Equation (3–54) is the same as Eq. (3–31) but has been obtained by a simpler procedure without manipulating position vectors.

EXAMPLE 3–8 Make a two-dimensional sketch of the equipotential lines and the electric field lines for an electric dipole.

Solution The equation of an equipotential surface of a charge distribution is obtained by setting the expression for V to equal a constant. Since q, d, and ϵ_0 in Eq. (3–53a) for an electric dipole are fixed quantities, a constant V requires a constant ratio ($\cos\theta/R^2$). Hence the equation for an equipotential surface is

$$R = c_V\sqrt{\cos\theta}, \tag{3-55}$$

where c_V is a constant. By plotting R versus θ for various values of c_V we draw the solid equipotential lines in Fig. 3–15. In the range $0 \le \theta \le \pi/2$, V is positive; R is maximum at $\theta = 0$ and zero at $\theta = 90°$. A mirror image is obtained in the range $\pi/2 \le \theta \le \pi$ where V is negative.

The electric field lines or streamlines represent the direction of the **E** field in space. We set

$$d\boldsymbol{\ell} = k\mathbf{E}, \tag{3-56}$$

where k is a constant. In spherical coordinates, Eq. (3–56) becomes (see Eq. 2–66)

$$\mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi = k(\mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta + \mathbf{a}_\phi E_\phi), \tag{3-57}$$

which can be written

$$\frac{dR}{E_R} = \frac{R\,d\theta}{E_\theta} = \frac{R\sin\theta\,d\phi}{E_\phi}. \tag{3-58}$$

For the electric dipole in Fig. 3–15 there is no E_ϕ component, and

$$\frac{dR}{2\cos\theta} = \frac{R\,d\theta}{\sin\theta}$$

or

$$\frac{dR}{R} = \frac{2\,d(\sin\theta)}{\sin\theta}. \tag{3-59}$$

Integrating Eq. (3–59), we obtain

$$R = c_E \sin^2\theta, \tag{3-60}$$

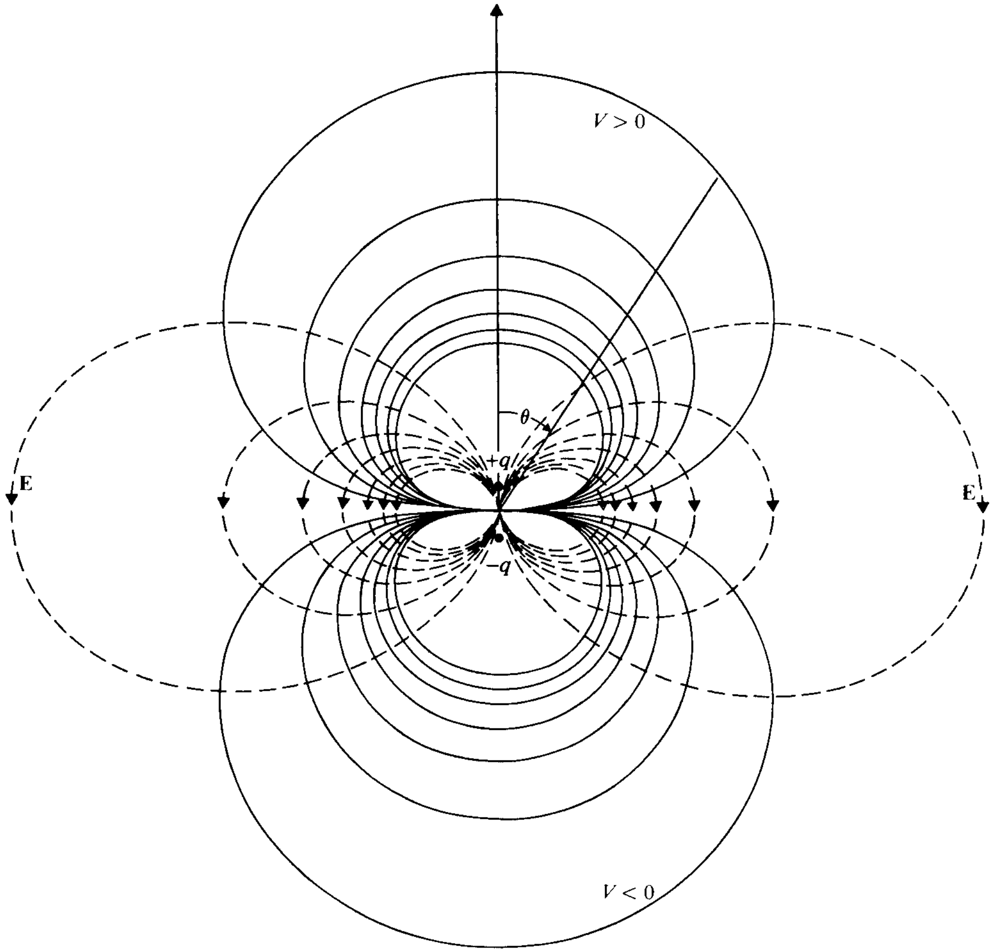


FIGURE 3–15
Equipotential and electric field lines of an electric dipole (Example 3–8).

where c_E is a constant. The electric field lines are drawn as dashed lines in Fig. 3–15. They are rotationally symmetrical about the z-axis (independent of ϕ) and are everywhere normal to the equipotential lines. ▬

The electric potential due to a continuous distribution of charge confined in a given region is obtained by integrating the contribution of an element of charge over the charged region. We have, for a volume charge distribution,

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv' \qquad \text{(V).}} \tag{3–61}$$

For a surface charge distribution,

$$V = \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\rho_s}{R}\, ds' \qquad \text{(V)}; \tag{3-62}$$

and for a line charge,

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_\ell}{R}\, d\ell' \qquad \text{(V)}. \tag{3-63}$$

We note here again that the integrals in Eqs. (3–61) and (3–62) represent integrations in three and two dimensions respectively.

EXAMPLE 3–9 Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries a uniform surface charge density ρ_s.

Solution Although the disk has circular symmetry, we cannot visualize a surface around it over which the normal component of **E** has a constant magnitude; hence Gauss's law is not useful for the solution of this problem. We use Eq. (3–62). Working with cylindrical coordinates indicated in Fig. 3–16, we have

$$ds' = r'\, dr'\, d\phi'$$

and

$$R = \sqrt{z^2 + r'^2}.$$

The electric potential at the point $P(0, 0, z)$ referring to the point at infinity is

$$\begin{aligned} V &= \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^b \frac{r'}{(z^2 + r'^2)^{1/2}}\, dr'\, d\phi' \\ &= \frac{\rho_s}{2\epsilon_0} [(z^2 + b^2)^{1/2} - |z|]. \end{aligned} \tag{3-64}$$

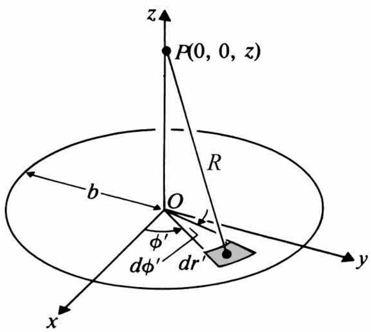


FIGURE 3–16
A uniformly charged disk (Example 3–9)

Therefore,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z \frac{\partial V}{\partial z}$$

$$= \begin{cases} \mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}\left[1 - z(z^2+b^2)^{-1/2}\right], & z > 0 \quad \text{(3–65a)} \\ -\mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}\left[1 + z(z^2+b^2)^{-1/2}\right], & z < 0. \quad \text{(3–65b)} \end{cases}$$

The determination of **E** field at an off-axis point would be a much more difficult problem. Do you know why?

For very large z, it is convenient to expand the second term in Eqs. (3–65a) and (3–65b) into a binomial series and neglect the second and all higher powers of the ratio (b^2/z^2). We have

$$z(z^2+b^2)^{-1/2} = \left(1 + \frac{b^2}{z^2}\right)^{-1/2} \cong 1 - \frac{b^2}{2z^2}.$$

Substituting this into Eqs. (3–65a) and (3–65b), we obtain

$$\mathbf{E} = \mathbf{a}_z \frac{(\pi b^2 \rho_s)}{4\pi\epsilon_0 z^2}$$

$$= \begin{cases} \mathbf{a}_z \dfrac{Q}{4\pi\epsilon_0 z^2}, & z > 0 \quad \text{(3–66a)} \\ -\mathbf{a}_z \dfrac{Q}{4\pi\epsilon_0 z^2}, & z < 0, \quad \text{(3–66b)} \end{cases}$$

where Q is the total charge on the disk. Hence, when the point of observation is very far away from the charged disk, the **E** field approximately follows the inverse square law as if the total charge were concentrated at a point. ▬

EXAMPLE 3–10 Obtain a formula for the electric field intensity along the axis of a uniform line charge of length L. The uniform line-charge density is ρ_ℓ.

Solution For an infinitely long line charge, the **E** field can be determined readily by applying Gauss's law, as in the solution to Example 3–5. However, for a line charge of finite length, as shown in Fig. 3–17, we cannot construct a Gaussian surface over which $\mathbf{E} \cdot d\mathbf{s}$ is constant. Gauss's law is therefore not useful here.

Instead, we use Eq. (3–63) by taking an element of charge $d\ell' = dz'$ at z'. The distance R from the charge element to the point $P(0, 0, z)$ along the axis of the line charge is

$$R = (z - z'), \qquad z > \frac{L}{2}.$$

Here it is extremely important to distinguish the position of the field point (unprimed coordinates) from the position of the source point (primed coordinates). We

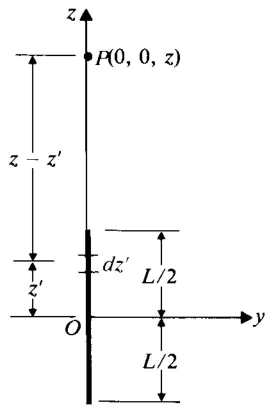


FIGURE 3–17
A finite line charge of a uniform line density ρ_ℓ (Example 3–10).

integrate over the source region:

$$\begin{aligned} V &= \frac{\rho_\ell}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dz'}{z - z'} \\ &= \frac{\rho_\ell}{4\pi\epsilon_0} \ln\left[\frac{z + (L/2)}{z - (L/2)}\right], \qquad z > \frac{L}{2}. \end{aligned} \tag{3–67}$$

The **E** field at P is the negative gradient of V with respect to the *unprimed* field coordinates. For this problem,

$$\mathbf{E} = -\mathbf{a}_z \frac{dV}{dz} = \mathbf{a}_z \frac{\rho_\ell L}{4\pi\epsilon_0[z^2 - (L/2)^2]}, \qquad z > \frac{L}{2}. \tag{3–68}$$

The preceding two examples illustrate the procedure for determining **E** by first finding V when Gauss's law cannot be conveniently applied. However, we emphasize that *if symmetry conditions exist such that a Gaussian surface can be constructed over which* $\mathbf{E} \cdot d\mathbf{s}$ *is constant, it is always easier to determine* **E** *directly*. The potential V, if desired, may be obtained from **E** by integration.

3–6 Conductors in Static Electric Field

So far we have discussed only the electric field of stationary charge distributions in free space or air. We now examine the field behavior in material media. In general, we classify materials according to their electrical properties into three types: ***conductors***, ***semiconductors***, and ***insulators*** (or ***dielectrics***). In terms of the crude atomic model of an atom consisting of a positively charged nucleus with orbiting electrons, the electrons in the outermost shells of the atoms of ***conductors*** are very loosely held and migrate easily from one atom to another. Most metals belong to this group. The electrons in the atoms of ***insulators*** or dielectrics, however, are confined to their orbits; they cannot be liberated in normal circumstances, even by the application of an external electric field. The electrical properties of ***semiconductors*** fall between those of conductors and insulators in that they possess a relatively small number of freely movable charges.

In terms of the band theory of solids we find that there are allowed energy bands for electrons, each band consisting of many closely spaced, discrete energy states. Between these energy bands there may be forbidden regions or gaps where no electrons of the solid's atom can reside. Conductors have an upper energy band partially filled with electrons or an upper pair of overlapping bands that are partially filled so that the electrons in these bands can move from one to another with only a small change in energy. Insulators or dielectrics are materials with a completely filled upper band, so conduction could not normally occur because of the existence of a large energy gap to the next higher band. If the energy gap of the forbidden region is relatively small, small amounts of external energy may be sufficient to excite the electrons in the filled upper band to jump into the next band, causing conduction. Such materials are semiconductors.

The macroscopic electrical property of a material medium is characterized by a constitutive parameter called ***conductivity***, which we will define in Chapter 5. The definition of conductivity is not important in this chapter because we are not dealing with current flow and are now interested only in the behavior of static electric fields in material media. In this section we examine the electric field and charge distribution both inside the bulk and on the surface of a conductor.

Assume for the present that some positive (or negative) charges are introduced in the interior of a conductor. An electric field will be set up in the conductor, the field exerting a force on the charges and making them move away from one another. This movement will continue until *all* the charges reach the conductor surface and redistribute themselves in such a way that both the charge and the field inside vanish. Hence,

Inside a Conductor (Under Static Conditions)	
$\rho = 0$	(3–69)
$\mathbf{E} = 0$	(3–70)

When there is no charge in the interior of a conductor ($\rho = 0$), $\mathbf{E}$ must be zero because, according to Gauss's law, the total outward electric flux through *any* closed surface constructed inside the conductor must vanish.

The charge distribution on the surface of a conductor depends on the shape of the surface. Obviously, the charges would not be in a state of equilibrium if there were a tangential component of the electric field intensity that produces a tangential

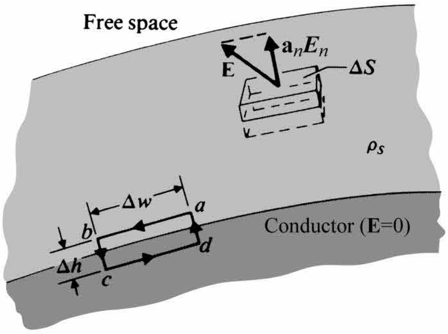


FIGURE 3–18
A conductor–free space interface.

force and moves the charges. Therefore, ***under static conditions the*** **E** ***field on a conductor surface is everywhere normal to the surface***. In other words, ***the surface of a conductor is an equipotential surface under static conditions***. As a matter of fact, since $\mathbf{E} = 0$ everywhere inside a conductor, the *whole* conductor has the same electrostatic potential. A finite time is required for the charges to redistribute on a conductor surface and reach the equilibrium state. This time depends on the conductivity of the material. For a good conductor such as copper this time is of the order of 10^{-19} (s), a very brief transient. (This point will be elaborated in Section 5–4.)

Figure 3–18 shows an interface between a conductor and free space. Consider the contour *abcda*, which has width $ab = cd = \Delta w$ and height $bc = da = \Delta h$. Sides *ab* and *cd* are parallel to the interface. Applying Eq. (3–8),† letting $\Delta h \to 0$, and noting that **E** in a conductor is zero, we obtain immediately

$$\oint_{abcda} \mathbf{E} \cdot d\ell = E_t \Delta w = 0$$

or

$$E_t = 0, \tag{3–71}$$

which says that ***the tangential component of the*** **E** ***field on a conductor surface is zero***. In order to find E_n, the normal component of **E** at the surface of the conductor, we construct a Gaussian surface in the form of a thin pillbox with the top face in free space and the bottom face in the conductor where $\mathbf{E} = 0$. Using Eq. (3–7), we obtain

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$$

or

$$E_n = \frac{\rho_s}{\epsilon_0}. \tag{3–72}$$

† We assume that Eqs. (3–7) and (3–8) are valid for regions containing discontinuous media.

Hence, ***the normal component of the E field at a conductor/free space boundary is equal to the surface charge density on the conductor divided by the permittivity of free space.*** Summarizing the *boundary conditions* at the conductor surface, we have

Boundary Conditions at a Conductor/Free Space Interface	
$E_t = 0$	(3–71)
$E_n = \dfrac{\rho_s}{\epsilon_0}$	(3–72)

When an uncharged conductor is placed in a static electric field, the external field will cause loosely held electrons inside the conductor to move in a direction opposite to that of the field and cause net positive charges to move in the direction of the field. These induced free charges will distribute on the conductor surface and create an *induced field* in such a way that they cancel the external field both inside the conductor and tangent to its surface. When the surface charge distribution reaches an equilibrium, all four relations, Eqs. (3–69) through (3–72), will hold; and the conductor is again an equipotential body.

EXAMPLE 3–11 A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine **E** and V as functions of the radial distance R.

Solution The geometry of the problem is shown in Fig. 3–19(a). Since there is spherical symmetry, it is simplest to use Gauss's law to determine **E** and then find V by integration. There are three distinct regions: (a) $R > R_o$, (b) $R_i < R < R_o$, and (c) $R < R_i$. Suitable spherical Gaussian surfaces will be constructed in these regions. Obviously, $\mathbf{E} = \mathbf{a}_R E_R$ in all three regions.

a) $R > R_o$ (Gaussian surface S_1):

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_{R1} 4\pi R^2 = \frac{Q}{\epsilon_0}$$

or

$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}. \tag{3–73}$$

The **E** field is the same as that of a point charge Q without the presence of the shell. The potential referring to the point at infinity is

$$V_1 = -\int_\infty^R (E_{R1})\, dR = \frac{Q}{4\pi\epsilon_0 R}. \tag{3–74}$$

b) $R_i < R < R_o$ (Gaussian surface S_2): Because of Eq. (3–70), we know that

$$E_{R2} = 0. \tag{3–75}$$

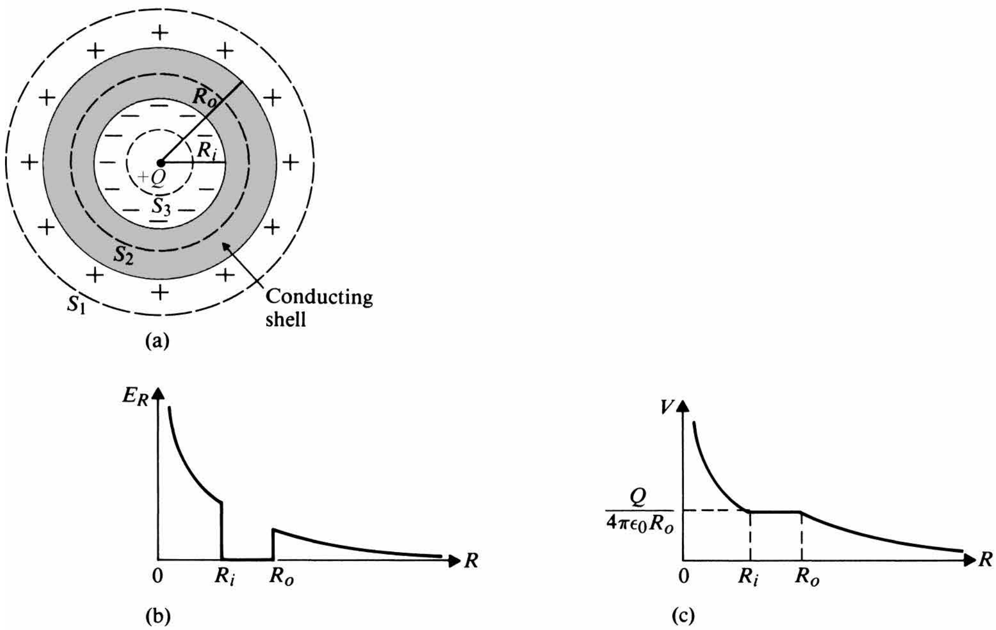


FIGURE 3–19
Electric field intensity and potential variations of a point charge $+Q$ at the center of a conducting shell (Example 3–11).

Since $\rho = 0$ in the conducting shell and since the total charge enclosed in surface S_2 must be zero, an amount of negative charge equal to $-Q$ must be induced on the inner shell surface at $R = R_i$. (This also means that an amount of positive charge equal to $+Q$ is induced on the outer shell surface at $R = R_o$.) The conducting shell is an equipotential body. Hence,

$$V_2 = V_1\Big|_{R=R_o} = \frac{Q}{4\pi\epsilon_0 R_o}. \tag{3–76}$$

c) $R < R_i$ (Gaussian surface S_3): Application of Gauss's law yields the same formula for E_{R3} as E_{R1} in Eq. (3–73) for the first region:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2}. \tag{3–77}$$

The potential in this region is

$$V_3 = -\int E_{R3}\,dR + C = \frac{Q}{4\pi\epsilon_0 R} + C,$$

where the integration constant C is determined by requiring V_3 at $R = R_i$ to equal V_2 in Eq. (3–76). We have

$$C = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_o} - \frac{1}{R_i}\right)$$

and

$$V_3 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i}\right). \tag{3–78}$$

The variations of E_R and V versus R in all three regions are plotted in Figs. 3–19(b) and 3–19(c). Note that while the electric intensity has discontinuous jumps, the potential remains continuous. A discontinuous jump in potential would mean an infinite electric field intensity. ■

3–7 Dielectrics in Static Electric Field

Ideal dielectrics do not contain free charges. When a dielectric body is placed in an external electric field, there are no induced free charges that move to the surface and make the interior charge density and electric field vanish, as with conductors. However, since dielectrics contain ***bound charges***, we cannot conclude that they have no effect on the electric field in which they are placed.

All material media are composed of atoms with a positively charged nucleus surrounded by negatively charged electrons. Although the molecules of dielectrics are macroscopically neutral, the presence of an external electric field causes a force to be exerted on each charged particle and results in small displacements of positive and negative charges in opposite directions. These displacements, though small in comparison to atomic dimensions, nevertheless *polarize* a dielectric material and create electric dipoles. The situation is depicted in Fig. 3–20. Inasmuch as electric dipoles do have nonvanishing electric potential and electric field intensity, we expect that the ***induced electric dipoles*** will modify the electric field both inside and outside the dielectric material.

The molecules of some dielectrics possess permanent dipole moments, even in the absence of an external polarizing field. Such molecules usually consist of two or

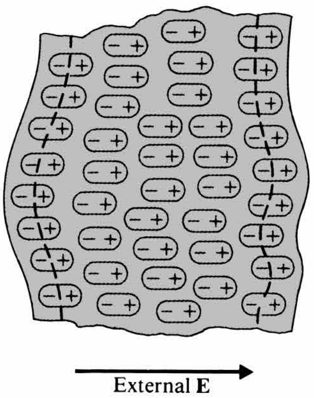


FIGURE 3–20
A cross section of a polarized dielectric medium.

more dissimilar atoms and are called ***polar molecules***, in contrast to ***nonpolar molecules***, which do not have permanent dipole moments. An example is the water molecule H_2O, which consists of two hydrogen atoms and one oxygen atom. The atoms do not arrange themselves in a manner that makes the molecule have a zero dipole moment; that is, the hydrogen atoms do not lie exactly on diametrically opposite sides of the oxygen atom.

The dipole moments of polar molecules are of the order of 10^{-30} (C·m). When there is no external field, the individual dipoles in a polar dielectric are randomly oriented, producing no net dipole moment macroscopically. An applied electric field will exert a torque on the individual dipoles and tend to align them with the field in a manner similar to that shown in Fig. 3–20.

Some dielectric materials can exhibit a permanent dipole moment even in the absence of an externally applied electric field. Such materials are called ***electrets***. Electrets can be made by heating (softening) certain waxes or plastics and placing them in an electric field. The polarized molecules in these materials tend to align with the applied field and to be frozen in their new positions after they return to normal temperatures. Permanent polarization remains without an external electric field. Electrets are the electrical equivalents of permanent magnets; they have found important applications in high fidelity electret microphones.[†]

3–7.1 EQUIVALENT CHARGE DISTRIBUTIONS OF POLARIZED DIELECTRICS

To analyze the macroscopic effect of induced dipoles we define a ***polarization vector***, **P**, as

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}), \tag{3-79}$$

where n is the number of molecules per unit volume and the numerator represents the vector sum of the induced dipole moments contained in a very small volume Δv. The vector **P**, a smoothed point function, is the *volume density of electric dipole moment*. The dipole moment $d\mathbf{p}$ of an elemental volume dv' is $d\mathbf{p} = \mathbf{P}\,dv'$, which produces an electrostatic potential (see Eq. 3–53b):

$$dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\, dv'. \tag{3-80}$$

Integrating over the volume V' of the dielectric, we obtain the potential due to the polarized dielectric.

[†] See, for instance, J. M. Crowley, *Fundamentals of Applied Electrostatics*, Section 8–3, *Wiley*, *New York*, 1986.

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2} \, dv', \tag{3–81}$$ [†]

where R is the distance from the elemental volume dv' to a fixed field point. In Cartesian coordinates,

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2, \tag{3–82}$$

and it is readily verified that the gradient of $1/R$ with respect to the *primed coordinates* is

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}. \tag{3–83}$$

Hence Eq. (3–81) can be written as

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'. \tag{3–84}$$

Recalling the vector identity (Problem 2–28),

$$\nabla' \cdot (f\mathbf{A}) = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f, \tag{3–85}$$

and letting $\mathbf{A} = \mathbf{P}$ and $f = 1/R$, we can rewrite Eq. (3–84) as

$$V = \frac{1}{4\pi\epsilon_0} \left[\int_{V'} \nabla' \cdot \left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\nabla' \cdot \mathbf{P}}{R} \, dv' \right]. \tag{3–86}$$

The first volume integral on the right side of Eq. (3–86) can be converted into a closed surface integral by the divergence theorem. We have

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R} \, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(-\nabla' \cdot \mathbf{P})}{R} \, dv', \tag{3–87}$$

where $\mathbf{a}'_n$ is the outward normal from the surface element ds' of the dielectric. Comparison of the two integrals on the right side of Eq. (3–87) with Eqs. (3–62) and (3–61), respectively, reveals that the electric potential (and therefore the electric field intensity also) due to a polarized dielectric may be calculated from the contributions of surface and volume charge distributions having, respectively, densities

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n} \tag{3–88}$$ [‡]

and

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.} \tag{3–89}$$ [‡]

[†] We note here that V on the left side of Eq. (3–81) represents the *electric potential* at a field point, and V' on the right side is the *volume* of the polarized dielectric.

[‡] The prime sign on $\mathbf{a}_n$ and ∇ has been dropped for simplicity, since Eqs. (3–88) and (3–89) involve only source coordinates and no confusion will result.

These are referred to as ***polarization charge densities*** or ***bound-charge densities***. In other words, a *polarized dielectric may be replaced by an equivalent polarization surface charge density* ρ_{ps} *and an equivalent polarization volume charge density* ρ_p *for field calculations:*

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{ps}}{R}\, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_p}{R}\, dv'. \tag{3-90}$$

Although Eqs. (3–88) and (3–89) were derived mathematically with the aid of a vector identity, a physical interpretation can be provided for the charge distributions. The sketch in Fig. 3–20 clearly indicates that charges from the ends of similarly oriented dipoles exist on surfaces not parallel to the direction of polarization. Consider an imaginary elemental surface Δs of a nonpolar dielectric. The application of an external electric field normal to Δs causes a separation d of the bound charges: positive charges $+q$ move a distance $d/2$ in the direction of the field, and negative charges $-q$ move an equal distance against the direction of the field. The net total charge ΔQ that crosses the surface Δs in the direction of the field is $nq\,d(\Delta s)$, where n is the number of molecules per unit volume. If the external field is not normal to Δs, the separation of the bound charges in the direction of $\mathbf{a}_n$ will be $\mathbf{d} \cdot \mathbf{a}_n$ and

$$\Delta Q = nq(\mathbf{d} \cdot \mathbf{a}_n)(\Delta s). \tag{3-91}$$

But $nq\mathbf{d}$, the dipole moment per unit volume, is by definition the polarization vector $\mathbf{P}$. We have

$$\Delta Q = \mathbf{P} \cdot \mathbf{a}_n(\Delta s) \tag{3-92}$$

and

$$\rho_{ps} = \frac{\Delta Q}{\Delta s} = \mathbf{P} \cdot \mathbf{a}_n,$$

as given in Eq. (3–88). Remember that $\mathbf{a}_n$ is always the *outward* normal. This relation correctly gives a positive surface charge on the right-hand surface in Fig. 3–20 and a negative surface charge on the left-hand surface.

For a surface S bounding a volume V, the net total charge flowing out of V as a result of polarization is obtained by integrating Eq. (3–92). The net charge *remaining* within the volume V is the *negative* of this integral:

$$\begin{aligned} Q &= -\oint_S \mathbf{P} \cdot \mathbf{a}_n\, ds \\ &= \int_V (-\nabla \cdot \mathbf{P})\, dv = \int_V \rho_p\, dv, \end{aligned} \tag{3-93}$$

which leads to the expression for the volume charge density in Eq. (3–89). Hence, when the divergence of $\mathbf{P}$ does not vanish, the bulk of the polarized dielectric appears to be charged. However, since we started with an electrically neutral dielectric body, the total charge of the body after polarization must remain zero. This can be readily verified by noting that

$$\begin{aligned}\text{Total charge} &= \oint_S \rho_{ps}\, ds + \int_V \rho_p\, dv \\ &= \oint_S \mathbf{P}\cdot\mathbf{a}_n\, ds - \int_V \nabla\cdot\mathbf{P}\, dv = 0,\end{aligned} \tag{3–94}$$

where the divergence theorem has again been applied.

3–8 Electric Flux Density and Dielectric Constant

Because a polarized dielectric gives rise to an equivalent volume charge density ρ_p, we expect the electric field intensity due to a given source distribution in a dielectric to be different from that in free space. In particular, the divergence postulated in Eq. (3–4) must be modified to include the effect of ρ_p; that is,

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p). \tag{3–95}$$

Using Eq. (3–89), we have

$$\nabla\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho. \tag{3–96}$$

We now define a new fundamental field quantity, the ***electric flux density***, or ***electric displacement***, **D**, such that

$$\boxed{\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \qquad (\text{C/m}^2).} \tag{3–97}$$

The use of the vector **D** enables us to write a divergence relation between the electric field and the distribution of *free charges* in any medium without the necessity of dealing explicitly with the polarization vector **P** or the polarization charge density ρ_p. Combining Eqs. (3–96) and (3–97), we obtain the new equation

$$\boxed{\nabla\cdot\mathbf{D} = \rho \qquad (\text{C/m}^3),} \tag{3–98}$$

where ρ is the volume density of *free charges*. Equations (3–98) and (3–5) are the two fundamental governing differential equations for electrostatics in any medium. Note that the permittivity of free space, ϵ_0, does not appear explicitly in these two equations.

The corresponding integral form of Eq. (3–98) is obtained by taking the volume integral of both sides. We have

$$\int_V \nabla\cdot\mathbf{D}\, dv = \int_V \rho\, dv \tag{3–99}$$

or

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{s} = Q \qquad (\text{C}).} \tag{3–100}$$

Equation (3–100), another form of ***Gauss's law***, states that ***the total outward flux of the electric displacement (or, simply, the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.*** As was indicated in Section 3–4, Gauss's law is most useful in determining the electric field due to charge distributions under symmetry conditions.

When the dielectric properties of the medium are *linear* and *isotropic*, the polarization is directly proportional to the electric field intensity, and the proportionality constant is independent of the direction of the field. We write

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \tag{3-101}$$

where χ_e is a dimensionless quantity called ***electric susceptibility***. A dielectric medium is linear if χ_e is independent of E and homogeneous if χ_e is independent of space coordinates. Substitution of Eq. (3–101) in Eq. (3–97) yields

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}),\end{aligned}} \tag{3-102}$$

where

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{3-103}$$

is a dimensionless quantity known as the ***relative permittivity*** or the ***dielectric constant*** of the medium. The coefficient $\epsilon = \epsilon_0 \epsilon_r$ is the ***absolute permittivity*** (often called simply ***permittivity***) of the medium and is measured in farads per meter (F/m). Air has a dielectric constant of 1.00059; hence its permittivity is usually taken as that of free space. The dielectric constants of some common materials are included in Table 3–1 on p. 114 and Appendix B–3.

Note that ϵ_r can be a function of space coordinates. If ϵ_r is independent of position, the medium is said to be ***homogenous***. A linear, homogeneous, and isotropic medium is called a ***simple medium***. The relative permittivity of a simple medium is a constant.

Later in the book we will learn that the effects of a lossy medium can be represented by a complex dielectric constant, whose imaginary part provides a measure of power loss in the medium and is, in general, frequency-dependent. For ***anisotropic*** materials the dielectric constant is different for different directions of the electric field, and **D** and **E** vectors generally have different directions; permittivity is a tensor. In matrix form we may write

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{3-104}$$

For crystals the reference coordinates can be chosen to be along the principal axes of the crystal so that the off-diagonal terms of the permittivity matrix in Eq. (3–104) are zero. We have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{3–105}$$

Media having the property represented by Eq. (3–105) are said to be ***biaxial***. We may write

$$D_x = \epsilon_1 E_x, \tag{3–106a}$$

$$D_y = \epsilon_2 E_y, \tag{3–106b}$$

$$D_z = \epsilon_3 E_z. \tag{3–106c}$$

If further, $\epsilon_1 = \epsilon_2$, then the medium is said to be ***uniaxial***. Of course, if $\epsilon_1 = \epsilon_2 = \epsilon_3$, we have an isotropic medium. We shall deal only with isotropic media in this book.

EXAMPLE 3–12 A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine **E**, V, **D**, and **P** as functions of the radial distance R.

Solution The geometry of this problem is the same as that of Example 3–11. The conducting shell has now been replaced by a dielectric shell, but the procedure of solution is similar. Because of the spherical symmetry, we apply Gauss's law to find **E** and **D** in three regions: (a) $R > R_o$; (b) $R_i < R < R_o$; and (c) $R < R_i$. Potential V is found from the negative line integral of **E**, and polarization **P** is determined by the relation

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \epsilon_0(\epsilon_r - 1)\mathbf{E}. \tag{3–107}$$

The **E**, **D**, and **P** vectors have only radial components. Refer to Fig. 3–21(a), where the Gaussian surfaces are not shown in order to avoid cluttering up the figure.

a) $R > R_o$
The situation in this region is exactly the same as that in Example 3–11. We have, from Eqs. (3–73) and (3–74),

$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$V_1 = \frac{Q}{4\pi\epsilon_0 R}.$$

From Eqs. (3–102) and (3–107) we obtain

$$D_{R1} = \epsilon_0 E_{R1} = \frac{Q}{4\pi R^2} \tag{3–108}$$

and

$$P_{R1} = 0. \tag{3–109}$$

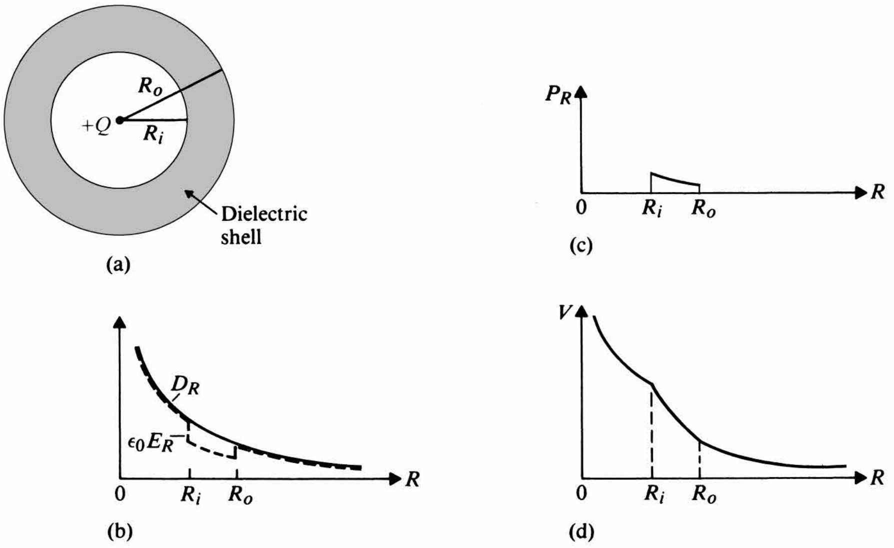


FIGURE 3–21
Field variations of a point charge $+Q$ at the center of a dielectric shell (Example 3–12).

b) $R_i < R < R_o$
The application of Gauss's law in this region gives us directly

$$E_{R2} = \frac{Q}{4\pi\epsilon_0\epsilon_r R^2} = \frac{Q}{4\pi\epsilon R^2}, \tag{3–110}$$

$$D_{R2} = \frac{Q}{4\pi R^2}, \tag{3–111}$$

$$P_{R2} = \left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R^2}. \tag{3–112}$$

Note that D_{R2} has the same expression as D_{R1} and that both E_R and P_R have a discontinuity at $R = R_o$. In this region,

$$\begin{aligned} V_2 &= -\int_\infty^{R_o} E_{R1}\,dR - \int_{R_o}^{R} E_{R2}\,dR \\ &= V_1\Big|_{R=R_o} - \frac{Q}{4\pi\epsilon}\int_{R_o}^{R}\frac{1}{R^2}\,dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1-\frac{1}{\epsilon_r}\right)\frac{1}{R_o} + \frac{1}{\epsilon_r R}\right]. \end{aligned} \tag{3–113}$$

c) $R < R_i$
Since the medium in this region is the same as that in the region $R > R_o$, the application of Gauss's law yields the same expressions for E_R, D_R, and P_R in

both regions:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2},$$

$$D_{R3} = \frac{Q}{4\pi R^2},$$

$$P_{R3} = 0.$$

To find V_3, we must add to V_2 at $R = R_i$ the negative line integral of E_{R3}:

$$\begin{aligned} V_3 &= V_2\Big|_{R=R_i} - \int_{R_i}^{R} E_{R3}\,dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]. \end{aligned} \tag{3–114}$$

The variations of $\epsilon_0 E_R$ and D_R versus R are plotted in Fig. 3–21(b). The difference $(D_R - \epsilon_0 E_R)$ is P_R and is shown in Fig. 3–21(c). The plot for V in Fig. 3–21(d) is a composite graph for V_1, V_2, and V_3 in the three regions. We note that D_R is a continuous curve exhibiting no sudden changes in going from one medium to another and that P_R exists only in the dielectric region. ■

It is instructive to compare Figs. 3–21(b) and 3–21(d) with Figs. 3–19(b) and 3–19(c), respectively, of Example 3–11. From Eqs. (3–88) and (3–89) we find

$$\begin{aligned} \rho_{ps}\Big|_{R=R_i} &= \mathbf{P}\cdot(-\mathbf{a}_R)\Big|_{R=R_i} = -P_{R2}\Big|_{R=R_i} \\ &= -\left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R_i^2} \end{aligned} \tag{3–115}$$

on the inner shell surface,

$$\begin{aligned} \rho_{ps}\Big|_{R=R_o} &= \mathbf{P}\cdot\mathbf{a}_R\Big|_{R=R_o} = P_{R2}\Big|_{R=R_o} \\ &= \left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R_o^2} \end{aligned} \tag{3–116}$$

on the outer shell surface, and

$$\begin{aligned} \rho_p &= -\nabla\cdot\mathbf{P} \\ &= -\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 P_{R2}) = 0. \end{aligned} \tag{3–117}$$

Equations (3–115), (3–116), and (3–117) indicate that there is no net polarization volume charge inside the dielectric shell. However, negative polarization surface charges exist on the inner surface and positive polarization surface charges on the outer surface. These surface charges produce an electric field intensity that is directed radially inward, thus reducing the $\mathbf{E}$ field in region 2 due to the point charge $+Q$ at the center.

TABLE 3–1
Dielectric Constants and Dielectric Strengths of Some Common Materials

Material	Dielectric Constant	Dielectric Strength (V/m)
Air (atmospheric pressure)	1.0	3×10^6
Mineral oil	2.3	15×10^6
Paper	2~4	15×10^6
Polystyrene	2.6	20×10^6
Rubber	2.3~4.0	25×10^6
Glass	4~10	30×10^6
Mica	6.0	200×10^6

3–8.1 DIELECTRIC STRENGTH

We have explained that an electric field causes small displacements of the bound charges in a dielectric material, resulting in polarization. If the electric field is very strong, it will pull electrons completely out of the molecules. The electrons will accelerate under the influence of the electric field, collide violently with the molecular lattice structure, and cause permanent dislocations and damage in the material. Avalanche effect of ionization due to collisions may occur. The material will become conducting, and large currents may result. This phenomenon is called a ***dielectric breakdown***. The maximum electric field intensity that a dielectric material can withstand without breakdown is the ***dielectric strength*** of the material. The approximate dielectric strengths of some common substances are given in Table 3–1. The dielectric strength of a material must not be confused with its dielectric constant.

A convenient number to remember is that the dielectric strength of air at the atmospheric pressure is 3 kV/mm. When the electric field intensity exceeds this value, air breaks down. Massive ionization takes place, and sparking (corona discharge) follows. Charge tends to concentrate at sharp points. In view of Eq. (3–72), the electric field intensity in the immediate vicinity of sharp points is much higher than that at points on a relatively flat surface with a small curvature. This is the principle upon which a lightning arrester with a sharp metal lightning rod on top of tall buildings works. When a cloud containing an abundance of electric charges approaches a tall building equipped with a lightning rod connected to the ground, charges of an opposite sign are attracted from the ground to the tip of the rod, where the electric field intensity is the strongest. As the electric field intensity exceeds the dielectric strength of the wet air, breakdown occurs, and the air near the tip is ionized and becomes conducting. The electric charges in the cloud are then discharged safely to the ground through the conducting path.

The fact that the electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature is illustrated quantitatively in the following example.

EXAMPLE 3–13 Consider two spherical conductors with radii b_1 and b_2 ($b_2 > b_1$) that are connected by a conducting wire. The distance of separation between the conductors is assumed to be very large in comparison to b_2 so that the charges on the spherical conductors may be considered as uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces.

Solution

a) Refer to Fig. 3–22. Since the spherical conductors are at the same potential, we have

$$\frac{Q_1}{4\pi\epsilon_0 b_1} = \frac{Q_2}{4\pi\epsilon_0 b_2}$$

or

$$\frac{Q_1}{Q_2} = \frac{b_1}{b_2}.$$

Hence the charges on the spheres are directly proportional to their radii. But, since

$$Q_1 + Q_2 = Q,$$

we find that

$$Q_1 = \frac{b_1}{b_1 + b_2} Q \quad \text{and} \quad Q_2 = \frac{b_2}{b_1 + b_2} Q.$$

b) The electric field intensities at the surfaces of the two conducting spheres are

$$E_{1n} = \frac{Q_1}{4\pi\epsilon_0 b_1^2} \quad \text{and} \quad E_{2n} = \frac{Q_2}{4\pi\epsilon_0 b_2^2},$$

so

$$\frac{E_{1n}}{E_{2n}} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1}.$$

The electric field intensities are therefore inversely proportional to the radii, being higher at the surface of the smaller sphere which has a larger curvature.

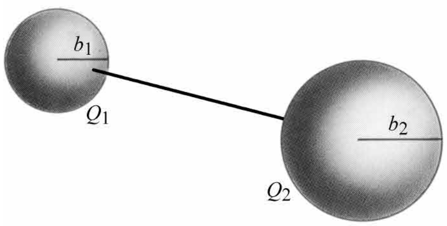


FIGURE 3–22
Two connected conducting spheres (Example 3–13).

3–9 Boundary Conditions for Electrostatic Fields

Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between two media. For instance, we may wish to determine how the **E** and **D** vectors change in crossing an interface. We already know the boundary conditions that must be satisfied at a conductor/free space interface. These conditions have been given in Eqs. (3–71) and (3–72). We now consider an interface between two general media shown in Fig. 3–23.

Let us construct a small path *abcda* with sides *ab* and *cd* in media 1 and 2, respectively, both being parallel to the interface and equal to Δw. Equation (3–8) is applied to this path. If we let sides $bc = da = \Delta h$ approach zero, their contributions to the line integral of **E** around the path can be neglected. We have

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

Therefore

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)},} \tag{3–118}$$

which states that ***the tangential component of an* E *field is continuous across an interface***. Eq. (3–118) simplifies to Eq. (3–71) if one of the media is a conductor. When media 1 and 2 are dielectrics with permittivities ϵ_1 and ϵ_2, respectively, we have

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}. \tag{3–119}$$

In order to find a relation between the normal components of the fields at a boundary, we construct a small pillbox with its top face in medium 1 and bottom

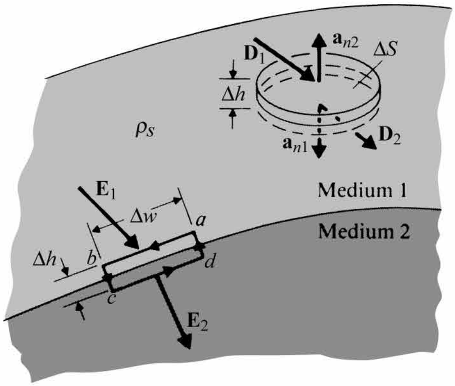


FIGURE 3–23
An interface between two media.

face in medium 2, as illustrated in Fig. 3–23. The faces have an area ΔS, and the height of the pillbox Δh is vanishingly small. Applying Gauss's law, Eq. (3–100), to the pillbox,[†] we have

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{s} &= (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S \\ &= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S \\ &= \rho_s \Delta S,\end{aligned} \tag{3–120}$$

where we have used the relation $\mathbf{a}_{n2} = -\mathbf{a}_{n1}$. Unit vectors $\mathbf{a}_{n1}$ and $\mathbf{a}_{n2}$ are, respectively, *outward* unit normals from media 1 and 2. From Eq. (3–120) we obtain

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s} \tag{3–121a}$$

or

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),} \tag{3–121b}$$

where the reference unit normal is outward from medium 2.

Eq. (3–121b) states that ***the normal component of* D *field is discontinuous across an interface where a surface charge exists—the amount of discontinuity being equal to the surface charge density***. If medium 2 is a conductor, $\mathbf{D}_2 = 0$ and Eq. (3–121b) becomes

$$D_{1n} = \epsilon_1 E_{1n} = \rho_s, \tag{3–122}$$

which simplifies to Eq. (3–72) when medium 1 is free space.

When two dielectrics are in contact with *no free charges* at the interface, $\rho_s = 0$, we have

$$D_{1n} = D_{2n} \tag{3–123}$$

or

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}. \tag{3–124}$$

Recapitulating, we find that the boundary conditions that must be satisfied for static electric fields are as follows:

Tangential components,	$E_{1t} = E_{2t}$;	(3–125)
Normal components,	$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$.	(3–126)

EXAMPLE 3–14 A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\mathbf{E}_o = \mathbf{a}_x E_o$ in free space. Determine $\mathbf{E}_i$, $\mathbf{D}_i$, and $\mathbf{P}_i$ inside the lucite.

[†] Equations (3–8) and (3–100) are assumed to hold for regions containing discontinuous media. See C. T. Tai, "On the presentation of Maxwell's theory," *Proceedings of the IEEE*, vol. 60, pp. 936–945, August 1972.

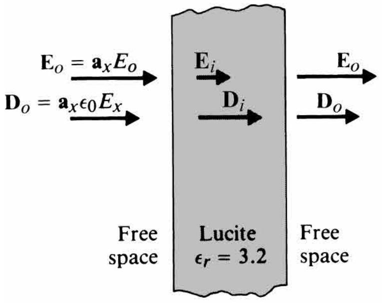


FIGURE 3–24
A lucite sheet in a uniform electric field (Example 3–14).

Solution We assume that the introduction of the lucite sheet does not disturb the original uniform electric field $\mathbf{E}_o$. The situation is depicted in Fig. 3–24. Since the interfaces are perpendicular to the electric field, only the normal field components need be considered. No free charges exist.

Boundary condition Eq. (3–123) at the left interface gives

$$\mathbf{D}_i = \mathbf{a}_x D_i = \mathbf{a}_x D_o$$

or

$$\mathbf{D}_i = \mathbf{a}_x \epsilon_0 E_o.$$

There is no change in electric flux density across the interface. The electric field intensity inside the lucite sheet is

$$\mathbf{E}_i = \frac{1}{\epsilon}\mathbf{D}_i = \frac{1}{\epsilon_o \epsilon_r}\mathbf{D}_i = \mathbf{a}_x \frac{E_o}{3.2}.$$

The polarization vector is zero outside the lucite sheet ($\mathbf{P}_o = 0$). Inside the sheet,

$$\begin{aligned}\mathbf{P}_i = \mathbf{D}_i - \epsilon_0 \mathbf{E}_i &= \mathbf{a}_x\left(1 - \frac{1}{3.2}\right)\epsilon_0 E_o \\ &= \mathbf{a}_x 0.6875\epsilon_0 E_o \qquad (\text{C/m}^2).\end{aligned}$$

■

Clearly, a similar application of the boundary condition Eq. (3–123) on the right interface will yield the original $\mathbf{E}_o$ and $\mathbf{D}_o$ in the free space on the right of the lucite sheet. Does the solution of this problem change if the original electric field is not uniform; that is, if $\mathbf{E}_o = \mathbf{a}_x E(y)$?

EXAMPLE 3–15 Two dielectric media with permittivities ϵ_1 and ϵ_2 are separated by a charge-free boundary as shown in Fig. 3–25. The electric field intensity in medium 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the electric field intensity at point P_2 in medium 2.

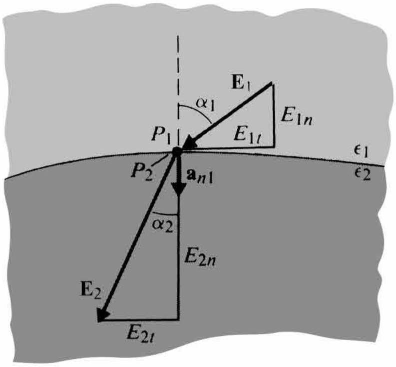


FIGURE 3–25
Boundary conditions at the interface between two dielectric media (Example 3–15).

Solution Two equations are needed to solve for two unknowns E_{2t} and E_{2n}. After E_{2t} and E_{2n} have been found, E_2 and α_2 will follow directly. Using Eqs. (3–118) and (3–123), we have

$$E_2 \sin \alpha_2 = E_1 \sin \alpha_1 \tag{3–127}$$

and

$$\epsilon_2 E_2 \cos \alpha_2 = \epsilon_1 E_1 \cos \alpha_1. \tag{3–128}$$

Division of Eq. (3–127) by Eq. (3–128) gives

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\epsilon_2}{\epsilon_1}.} \tag{3–129}$$

The magnitude of $\mathbf{E}_2$ is

$$E_2 = \sqrt{E_{2t}^2 + E_{2n}^2} = \sqrt{(E_2 \sin \alpha_2)^2 + (E_2 \cos \alpha_2)^2}$$
$$= \left[(E_1 \sin \alpha_1)^2 + \left(\frac{\epsilon_1}{\epsilon_2} E_1 \cos \alpha_1\right)^2\right]^{1/2}$$

or

$$\boxed{E_2 = E_1\left[\sin^2 \alpha_1 + \left(\frac{\epsilon_1}{\epsilon_2} \cos \alpha_1\right)^2\right]^{1/2}.} \tag{3–130}$$

By examining Fig. 3–25, can you tell whether ϵ_1 is larger or smaller than ϵ_2? ■

EXAMPLE 3–16 When a coaxial cable is used to carry electric power, the radius of the inner conductor is determined by the load current, and the overall size by the voltage and the type of insulating material used. Assume that the radius of the inner conductor is 0.4 (cm) and that concentric layers of rubber ($\epsilon_{rr} = 3.2$) and polystyrene ($\epsilon_{rp} = 2.6$) are used as insulating materials. Design a cable that is to work at a voltage rating of 20 (kV). In order to avoid breakdown due to voltage surges caused by lightning and other abnormal external conditions, the maximum electric field intensities in the insulating materials are not to exceed 25% of their dielectric strengths.

Solution From Table 3–1, p. 114, we find the dielectric strengths of rubber and polystyrene to be 25 × 10⁶ (V/m) and 20 × 10⁶ (V/m), respectively. Using Eq. (3–40) for specified 25% of dielectric strengths, we have the following.

$$\text{In rubber:} \qquad \text{Max } E_r = 0.25 \times 25 \times 10^6 = \frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{3.2r_i}\right). \tag{3–131a}$$

$$\text{In polystyrene:} \qquad \text{Max } E_p = 0.25 \times 20 \times 10^6 = \frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{2.6r_p}\right). \tag{3–131b}$$

Combination of Eqs. (3–131a) and (3–131b) yields

$$r_p = 1.54r_i = 0.616 \qquad \text{(cm)}. \tag{3–132}$$

Equation (3–132) indicates that the insulating layer of polystyrene should be placed outside of that of rubber, as shown in Fig. 3–26(a). (It would be interesting to determine what would happen if the polystyrene layer were placed inside the rubber layer.)

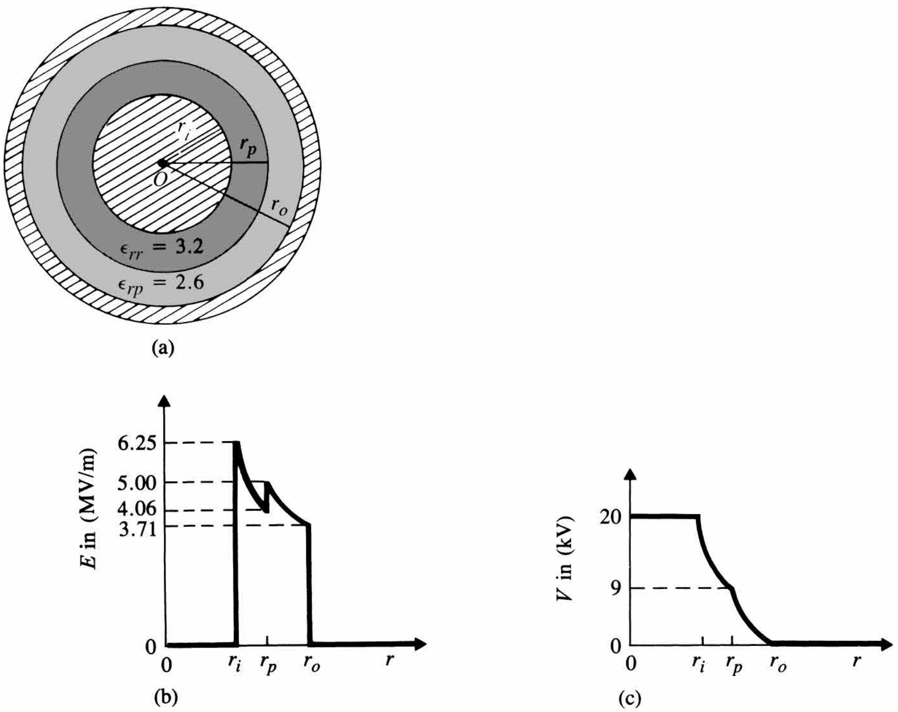


FIGURE 3–26
Cross section of coaxial cable with two different kinds of insulating material (Example 3–16).

The cable is to work at a potential difference of 20,000 (V) between the inner and outer conductors. We set

$$-\int_{r_o}^{r_p} E_p\,dr - \int_{r_p}^{r_i} E_r\,dr = 20{,}000,$$

where both E_p and E_r have the form given in Eq. (3–40). The above relation leads to

$$\frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{\epsilon_{rp}}\ln\frac{r_o}{r_p} + \frac{1}{\epsilon_{rr}}\ln\frac{r_p}{r_i}\right) = 20{,}000$$

or

$$\frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{2.6}\ln\frac{r_o}{1.54r_i} + \frac{1}{3.2}\ln 1.54\right) = 20{,}000. \tag{3–133}$$

Since $r_i = 0.4$ (cm) is given, r_o can be determined by finding the factor $\rho_\ell/2\pi\epsilon_0$ from Eq. (3–131a) and then using it in Eq. (3–133). We obtain $\rho_\ell/2\pi\epsilon_0 = 8 \times 10^4$, and $r_o = 2.08r_i = 0.832$ (cm).

In Figs. 3–26(b) and 3–26(c) are plotted the variations of the radial electric field intensity E and the potential V referred to that of the outer sheath. Note that E has discontinuous jumps, while the V curve is continuous. The reader should verify all the indicated numerical values. ∎

3–10 Capacitance and Capacitors

From Section 3–6 we understand that a conductor in a static electric field is an equipotential body and that charges deposited on a conductor will distribute themselves on its surface in such a way that the electric field inside vanishes. Suppose the potential due to a charge Q is V. Obviously, increasing the total charge by some factor k would merely increase the surface charge density ρ_s everywhere by the same factor without affecting the charge distribution because the conductor remains an equipotential body in a static situation. We may conclude from Eq. (3–62) that the potential of an isolated conductor is directly proportional to the total charge on it. This may also be seen from the fact that increasing V by a factor of k increases $\mathbf{E} = -\nabla V$ by a factor of k. But from Eq. (3–72), $\mathbf{E} = \mathbf{a}_n\rho_s/\epsilon_0$; it follows that ρ_s, and consequently the total charge Q will also increase by a factor of k. The ratio Q/V therefore remains unchanged. We write

$$\boxed{Q = CV,} \tag{3–134}$$

where the constant of proportionality C is called the **capacitance** of the isolated conducting body. The capacitance is the electric charge that must be added to the body per unit increase in its electric potential. Its SI unit is coulomb per volt, or farad (F).

Of considerable importance in practice is the **capacitor**, which consists of two conductors separated by free space or a dielectric medium. The conductors may be of arbitrary shapes as in Fig. 3–27. When a d-c voltage source is connected between the conductors, a charge transfer occurs, resulting in a charge $+Q$ on one conductor

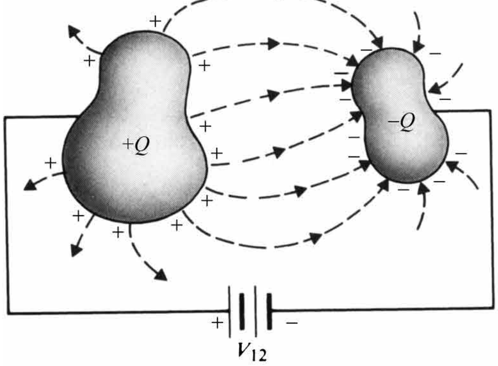


FIGURE 3–27
A two-conductor capacitor.

and $-Q$ on the other. Several electric field lines originating from positive charges and terminating on negative charges are shown in Fig. 3–27. Note that the field lines are perpendicular to the conductor surfaces, which are equipotential surfaces. Equation (3–134) applies here if V is taken to mean the potential difference between the two conductors, V_{12}. That is,

$$\boxed{C = \frac{Q}{V_{12}} \quad \text{(F)}.} \tag{3–135}$$

The capacitance of a capacitor is a physical property of the two-conductor system. It depends on the geometry of the conductors and on the permittivity of the medium between them; it does *not* depend on either the charge Q or the potential difference V_{12}. A capacitor has a capacitance even when no voltage is applied to it and no free charges exist on its conductors. Capacitance C can be determined from Eq. (3–135) by either (1) assuming a V_{12} and determining Q in terms of V_{12}, or (2) assuming a Q and determining V_{12} in terms of Q. At this stage, since we have not yet studied the methods for solving boundary-value problems (which will be taken up in Chapter 4), we find C by the second method. The procedure is as follows:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume charges $+Q$ and $-Q$ on the conductors.
3. Find $\mathbf{E}$ from Q by Eq. (3–122), Gauss's law, or other relations.
4. Find V_{12} by evaluating
$$V_{12} = -\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}$$
from the conductor carrying $-Q$ to the other carrying $+Q$.
5. Find C by taking the ratio Q/V_{12}.

EXAMPLE 3–17 A parallel-plate capacitor consists of two parallel conducting plates of area S separated by a uniform distance d. The space between the plates is filled with a dielectric of a constant permittivity ϵ. Determine the capacitance.

Solution A cross section of the capacitor is shown in Fig. 3–28. It is obvious that the appropriate coordinate system to use is the Cartesian coordinate system. Following the procedure outlined above, we put charges $+Q$ and $-Q$ on the upper and lower conducting plates, respectively. The charges are assumed to be uniformly distributed over the conducting plates with surface densities $+\rho_s$ and $-\rho_s$, where

$$\rho_s = \frac{Q}{S}.$$

From Eq. (3–122) we have

$$\mathbf{E} = -\mathbf{a}_y \frac{\rho_s}{\epsilon} = -\mathbf{a}_y \frac{Q}{\epsilon S},$$

which is constant within the dielectric if the fringing of the electric field at the edges of the plates is neglected. Now

$$V_{12} = -\int_{y=0}^{y=d} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_0^d \left(-\mathbf{a}_y \frac{Q}{\epsilon S}\right) \cdot (\mathbf{a}_y\, dy) = \frac{Q}{\epsilon S} d.$$

Therefore, *for a parallel-plate capacitor,*

$$\boxed{C = \frac{Q}{V_{12}} = \epsilon \frac{S}{d},} \tag{3–136}$$

which is independent of Q or V_{12}.

For this problem we could have started by assuming a potential difference V_{12} between the upper and lower plates. The electric field intensity between the plates is uniform and equals

$$\mathbf{E} = -\mathbf{a}_y \frac{V_{12}}{d}.$$

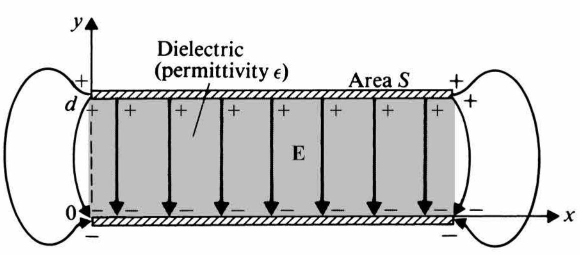


FIGURE 3–28
Cross section of a parallel-plate capacitor (Example 3–17).

The surface charge densities at the upper and lower conducting plates are $+\rho_s$ and $-\rho_s$, respectively, where, in view of Eq. (3–72),

$$\rho_s = \epsilon E_y = \epsilon \frac{V_{12}}{d}.$$

Therefore, $Q = \rho_s S = (\epsilon S/d)V_{12}$ and $C = Q/V_{12} = \epsilon S/d$, as before.

EXAMPLE 3–18 A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ϵ, and the length of the capacitor is L. Determine the capacitance of this capacitor.

Solution We use cylindrical coordinates for this problem. First we assume charges $+Q$ and $-Q$ on the surface of the inner conductor and the inner surface of the outer conductor, respectively. The **E** field in the dielectric can be obtained by applying Gauss's law to a cylindrical Gaussian surface within the dielectric $a < r < b$. (Note that Eq. (3–122) gives only the *normal component* of the **E** field *at* a conductor surface. Since the conductor surfaces are not planes here, the **E** field is not constant in the dielectric and Eq. (3–122) cannot be used to find **E** in the $a < r < b$ region.) Referring to Fig. 3–29 and applying Gauss's law, we have

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}. \tag{3–137}$$

Again we neglect the fringing effect of the field near the edges of the conductors. The potential difference between the inner and outer conductors is

$$\begin{aligned} V_{ab} &= -\int_{r=b}^{r=a} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_b^a \left(\mathbf{a}_r \frac{Q}{2\pi\epsilon L r}\right) \cdot (\mathbf{a}_r\, dr) \\ &= \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right). \end{aligned} \tag{3–138}$$

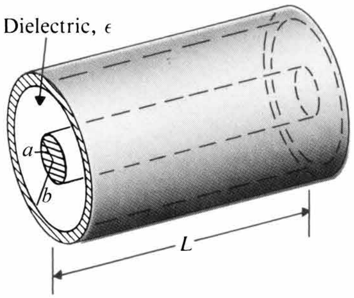


FIGURE 3–29
A cylindrical capacitor (Example 3–18).

Therefore, *for a cylindrical capacitor,*

$$\boxed{C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}.} \tag{3–139}$$

We could not solve this problem from an assumed V_{ab} because the electric field is not uniform between the inner and outer conductors. Thus we would not know how to express **E** and Q in terms of V_{ab} until we learned how to solve such a boundary-value problem. ■

EXAMPLE 3–19 A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a spherical inner wall of radius R_o. The space in between is filled with a dielectric of permittivity ϵ. Determine the capacitance.

Solution Assume charges $+Q$ and $-Q$ on the inner and outer conductors, respectively, of the spherical capacitor in Fig. 3–30. Applying Gauss's law to a spherical Gaussian surface with radius $R(R_i < R < R_o)$, we have

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{Q}{4\pi\epsilon R^2}$$

$$V = -\int_{R_o}^{R_i} \mathbf{E} \cdot (\mathbf{a}_R\, dR) = -\int_{R_o}^{R_i} \frac{Q}{4\pi\epsilon R^2}\, dR = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right).$$

Therefore, *for a spherical capacitor,*

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\epsilon}{\frac{1}{R_i} - \frac{1}{R_o}}.} \tag{3–140}$$

■

For an isolated conducting sphere of a radius R_i, $R_o \to \infty$, $C = 4\pi\epsilon R_i$.

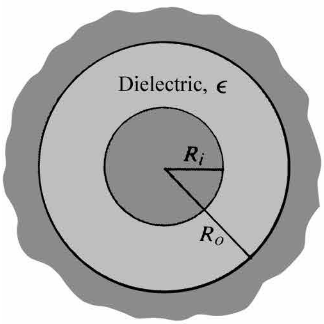


FIGURE 3–30
A spherical capacitor (Example 3–19).

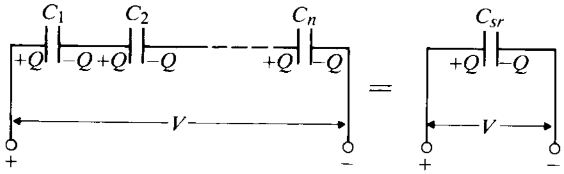


FIGURE 3–31
Series connection of capacitors.

3–10.1 SERIES AND PARALLEL CONNECTIONS OF CAPACITORS

Capacitors are often combined in various ways in electric circuits. The two basic ways are series and parallel connections. In the series, or head-to-tail, connection shown in Fig. 3–31,[†] the external terminals are from the first and last capacitors only. When a potential difference or electrostatic voltage V is applied, charge cumulations on the conductors connected to the external terminals are $+Q$ and $-Q$. Charges will be induced on the internally connected conductors such that $+Q$ and $-Q$ will appear on each capacitor independently of its capacitance. The potential differences across the individual capacitors are Q/C_1, $Q/C_2, \ldots, Q/C_n$, and

$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n},$$

where C_{sr} is the equivalent capacitance of the series-connected capacitors. We have

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.} \tag{3–141}$$

In the parallel connection of capacitors the external terminals are connected to the conductors of all the capacitors as in Fig. 3–32. When a potential difference V is applied to the terminals, the charge cumulated on a capacitor depends on its capacitance. The total charge is the sum of all the charges.

$$\begin{aligned} Q &= Q_1 + Q_2 + \cdots + Q_n \\ &= C_1V + C_2V + \cdots + C_nV = C_{||}V \end{aligned}$$

Therefore, the equivalent capacitance of the parallel-connected capacitors is

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.} \tag{3–142}$$

[†] Capacitors, whatever their actual shape, are conventionally represented in circuits by pairs of parallel bars.

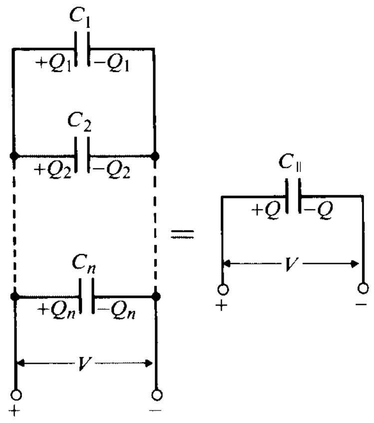


FIGURE 3–32
Parallel connection of capacitors.

We note that the formula for the equivalent capacitance of series-connected capacitors is similar to that for the equivalent resistance of parallel-connected resistors and that the formula for the equivalent capacitance of parallel-connected capacitors is similar to that for the equivalent resistance of series-connected resistors. Can you explain this?

EXAMPLE 3–20 Four capacitors $C_1 = 1\ (\mu F)$, $C_2 = 2\ (\mu F)$, $C_3 = 3\ (\mu F)$, and $C_4 = 4\ (\mu F)$ are connected as in Fig. 3–33. A d-c voltage of 100 (V) is applied to the external terminals a–b. Determine the following: (a) the total equivalent capacitance between terminals a–b, (b) the charge on each capacitor, and (c) the potential difference across each capacitor.

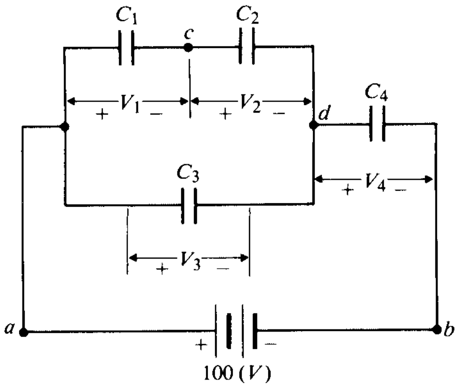


FIGURE 3–33
A combination of capacitors (Example 3–20).

Solution

a) The equivalent capacitance C_{12} of C_1 and C_2 in series is

$$C_{12} = \frac{1}{(1/C_1) + (1/C_2)} = \frac{C_1 C_2}{C_1 + C_2} = \frac{2}{3} \quad (\mu\text{F}).$$

The combination of C_{12} in parallel with C_3 gives

$$C_{123} = C_{12} + C_3 = \tfrac{11}{3} \quad (\mu\text{F}).$$

The total equivalent capacitance C_{ab} is then

$$C_{ab} = \frac{C_{123} C_4}{C_{123} + C_4} = \frac{44}{23} = 1.913 \quad (\mu\text{F}).$$

b) Since the capacitances are given, the voltages can be found as soon as the charges have been determined. We have four unknowns: Q_1, Q_2, Q_3, and Q_4. Four equations are needed for their determination.

Series connection of C_1 and C_2: $$Q_1 = Q_2.$$

Kirchhoff's voltage law, $V_1 + V_2 = V_3$: $$\frac{Q_1}{C_1} + \frac{Q_2}{C_2} = \frac{Q_3}{C_3}.$$

Kirchhoff's voltage law, $V_3 + V_4 = 100$: $$\frac{Q_3}{C_3} + \frac{Q_4}{C_4} = 100.$$

Series connection at d: $$Q_2 + Q_3 = Q_4.$$

Using the given values of C_1, C_2, C_3, and C_4 and solving the equations, we obtain

$$Q_1 = Q_2 = \frac{800}{23} = 34.8 \quad (\mu\text{C}),$$

$$Q_3 = \frac{3600}{23} = 156.5 \quad (\mu\text{C}),$$

$$Q_4 = \frac{4400}{23} = 191.3 \quad (\mu\text{C}).$$

c) Dividing the charges by the capacitances, we find

$$V_1 = \frac{Q_1}{C_1} = 34.8 \quad (\text{V}),$$

$$V_2 = \frac{Q_2}{C_2} = 17.4 \quad (\text{V}),$$

$$V_3 = \frac{Q_3}{C_3} = 52.2 \quad (\text{V}),$$

$$V_4 = \frac{Q_4}{C_4} = 47.8 \quad (\text{V}).$$

These results can be checked by verifying that $V_1 + V_2 = V_3$ and that $V_3 + V_4 = 100$ (V). ■

3–10.2 CAPACITANCES IN MULTICONDUCTOR SYSTEMS

We now consider the situation of more than two conducting bodies in an isolated system, such as that shown in Fig. 3–34. The positions of the conductors are arbitrary, and one of the conductors may represent the ground. Obviously, the presence of a charge on any one of the conductors will affect the potential of all the others. Since the relation between potential and charge is linear, we may write the following set of N equations relating the potentials $V_1, V_2, \ldots, V_N$ of the N conductors to the charges $Q_1, Q_2, \ldots, Q_N$:

$$\begin{aligned} V_1 &= p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N, \\ V_2 &= p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N, \\ &\vdots \\ V_N &= p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N. \end{aligned} \tag{3–143}$$

In Eqs. (3–143) the p_{ij}'s are called the ***coefficients of potential***, which are constants whose values depend on the shape and position of the conductors as well as the permittivity of the surrounding medium. We note that in an isolated system,

$$Q_1 + Q_2 + Q_3 + \cdots + Q_N = 0. \tag{3–144}$$

The N linear equations in (3–143) can be inverted to express the charges as functions of potentials as follows:

$$\begin{aligned} Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\ Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\ &\vdots \\ Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N, \end{aligned} \tag{3–145}$$

where the c_{ij}'s are constants whose values depend only on the p_{ij}'s in Eqs. (3–143). The coefficients c_{ii}'s are called the ***coefficients of capacitance***, which equal the ratios of the charge Q_i on and the potential V_i of the ith conductor ($i = 1, 2, \ldots, N$) with all other conductors grounded. The c_{ij}'s ($i \neq j$) are called the ***coefficients of induction.*** If a positive Q_i exists on the ith conductor, V_i will be positive, but the charge Q_j

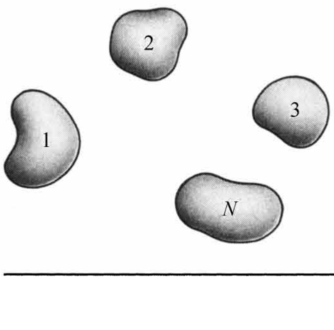


FIGURE 3–34
A multiconductor system.

induced on the jth ($j \neq i$) conductor will be negative. Hence the coefficients of capacitance c_{ii} are positive, and the coefficients of induction c_{ij} are negative. The condition of reciprocity guarantees that $p_{ij} = p_{ji}$ and $c_{ij} = c_{ji}$.

To establish a physical meaning to the coefficients of capacitance and the coefficients of induction, let us consider a four-conductor system as depicted in Fig. 3–34 with the stipulation that the conductor labeled N is now the conducting earth at zero potential and is designated by the number 0. A schematic diagram of the four-conductor system is shown in Fig. 3–35, in which the conductors 1, 2, and 3 have been drawn as simple dots (nodes). Coupling capacitances have been shown between pairs of nodes and between the three nodes and the ground. If Q_1, Q_2, Q_3 and V_1, V_2, V_3 denote the charges and the potentials, respectively, of conductors 1, 2, and 3, the first three equations in (3–145) become

$$Q_1 = c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \quad (3\text{–}146\text{a})$$

$$Q_2 = c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \quad (3\text{–}146\text{b})$$

$$Q_3 = c_{13}V_1 + c_{23}V_2 + c_{33}V_3, \quad (3\text{–}146\text{c})$$

where we have used the symmetry relation $c_{ij} = c_{ji}$. On the other hand, we can write another set of three $Q \sim V$ relations based on the schematic diagram in Fig. 3–35:

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3), \quad (3\text{–}147\text{a})$$

$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3), \quad (3\text{–}147\text{b})$$

$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2), \quad (3\text{–}147\text{c})$$

where C_{10}, C_{20}, and C_{30} are self-partial capacitances and C_{ij} ($i \neq j$) are mutual partial capacitances.

Equations (3–147a), (3–147b), and (3–147c) can be rearranged as

$$Q_1 = (C_{10} + C_{12} + C_{13})V_1 - C_{12}V_2 - C_{13}V_3, \quad (3\text{–}148\text{a})$$

$$Q_2 = -C_{12}V_1 + (C_{20} + C_{12} + C_{23})V_2 - C_{23}V_3, \quad (3\text{–}148\text{b})$$

$$Q_3 = -C_{13}V_1 - C_{23}V_2 + (C_{30} + C_{13} + C_{23})V_3. \quad (3\text{–}148\text{c})$$

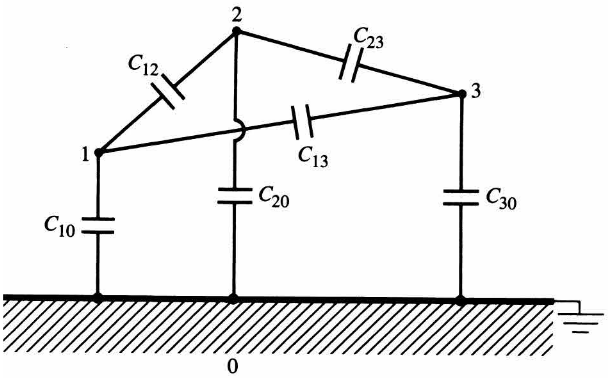


FIGURE 3–35
Schematic diagram of three conductors and the ground.

Comparing Eqs. (3–148) with Eqs. (3–146), we obtain

$$c_{11} = C_{10} + C_{12} + C_{13}, \tag{3–149a}$$
$$c_{22} = C_{20} + C_{12} + C_{23}, \tag{3–149b}$$
$$c_{33} = C_{30} + C_{13} + C_{23}, \tag{3–149c}$$

and

$$c_{12} = -C_{12}, \tag{3–150a}$$
$$c_{23} = -C_{23}, \tag{3–150b}$$
$$c_{13} = -C_{13}. \tag{3–150c}$$

On the basis of Eq. (3–149a) we can interpret the coefficient of capacitance c_{11} as the total capacitance between conductor 1 and all the other conductors connected together to ground; similarly for c_{22} and c_{33}. Equations (3–150) indicate that the coefficients of inductances are the negative of the mutual partial capacitances. Inverting Eqs. (3–149), we can express the conductor-to-ground capacitances in terms of the coefficients of capacitance and coefficients of induction:

$$C_{10} = c_{11} + c_{12} + c_{13}, \tag{3–151a}$$
$$C_{20} = c_{22} + c_{12} + c_{23}, \tag{3–151b}$$
$$C_{30} = c_{33} + c_{13} + c_{23}. \tag{3–151c}$$

EXAMPLE 3–21 Three horizontal parallel conducting wires, each of radius a and isolated from the ground, are separated from one another as shown in Fig. 3–36. Assuming $d \gg a$, determine the partial capacitances per unit length between the wires.

Solution We designate the wires as conductors 0, 1, and 2, as indicated in Fig. 3–36. Choosing conductor 0 as the reference and using Eq. (3–138), we can write two equations for the potential differences V_{10} and V_{20} due to the three wires as follows:

$$V_{10} = \frac{\rho_{\ell 0}}{2\pi\epsilon_0} \ln \frac{a}{d} + \frac{\rho_{\ell 1}}{2\pi\epsilon_0} \ln \frac{d}{a} + \frac{\rho_{\ell 2}}{2\pi\epsilon_0} \ln \frac{3d}{2d}$$

or

$$2\pi\epsilon_0 V_{10} = \rho_{\ell 0} \ln \frac{a}{d} + \rho_{\ell 1} \ln \frac{d}{a} + \rho_{\ell 2} \ln \frac{3}{2}, \tag{3–152a}$$

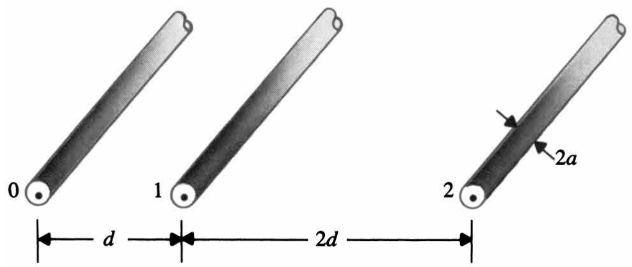


FIGURE 3–36
Three parallel wires (Example 3–21).

where $\rho_{\ell 0}$, $\rho_{\ell 1}$, and $\rho_{\ell 2}$ denote the charges per unit length on wires 0, 1, and 2 respectively. Similarly,

$$2\pi\epsilon_0 V_{20} = \rho_{\ell 0} \ln \frac{a}{3d} + \rho_{\ell 1} \ln \frac{d}{2d} + \rho_{\ell 2} \ln \frac{3d}{a}. \tag{3-152b}$$

For the isolated system of three conductors we have $\rho_{\ell 0} + \rho_{\ell 1} + \rho_{\ell 2} = 0$, or

$$\rho_{\ell 0} = -(\rho_{\ell 1} + \rho_{\ell 2}). \tag{3-153}$$

Combination of Eqs. (3–152a), (3–152b), and (3–153) yields

$$2\pi\epsilon_0 V_{10} = \rho_{\ell 1} 2 \ln \frac{d}{a} + \rho_{\ell 2} \ln \frac{3d}{2a}, \tag{3-154a}$$

$$2\pi\epsilon_0 V_{20} = \rho_{\ell 1} \ln \frac{3d}{2a} + \rho_{\ell 2} 2 \ln \frac{3d}{a}. \tag{3-154b}$$

Equations (3–154a) and (3–154b) can be used to solve for $\rho_{\ell 1}$ and $\rho_{\ell 2}$ as functions of V_{10} and V_{20}.

$$\rho_{\ell 1} = \Delta_0\left(V_{10} 2 \ln \frac{3d}{a} - V_{20} \ln \frac{3d}{2a}\right), \tag{3-155a}$$

$$\rho_{\ell 2} = \Delta_0\left(-V_{10} \ln \frac{3d}{2a} + V_{20} 2 \ln \frac{d}{a}\right), \tag{3-155b}$$

where

$$\Delta_0 = \frac{2\pi\epsilon_0}{4 \ln \frac{d}{a} \ln \frac{3d}{a} - \left(\ln \frac{3d}{2a}\right)^2}. \tag{3-156}$$

Comparing Eqs. (3–155) with Eqs. (3–146), (3–148), and (3–151), we obtain the following partial capacitances per unit length for the given three-wire system:

$$C_{12} = -c_{12} = \Delta_0 \ln \frac{3d}{2a}, \tag{3-157a}$$

$$C_{10} = c_{11} + c_{12} = \Delta_0\left(2 \ln \frac{3d}{a} - \ln \frac{3d}{2a}\right), \tag{3-157b}$$

$$C_{20} = c_{22} + c_{12} = \Delta_0\left(2 \ln \frac{d}{a} - \ln \frac{3d}{2a}\right). \tag{3-157c}$$

■

3–10.3 ELECTROSTATIC SHIELDING

Electrostatic shielding, a technique for reducing capacitive coupling between conducting bodies, is important in some practical applications. Let us consider the situation shown in Fig. 3–37, in which a grounded conducting shell 2 completely

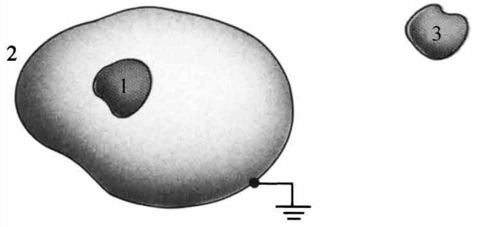


FIGURE 3–37
Illustrating electrostatic shielding.

encloses conducting body 1. Setting $V_2 = 0$ in Eq. (3–147a), we have

$$Q_1 = C_{10}V_1 + C_{12}V_1 + C_{13}(V_1 - V_3). \tag{3–158}$$

When $Q_1 = 0$, there is no field inside shell 2; hence body 1 and shell 2 have the same potential, $V_1 = V_2 = 0$. From Eq. (3–158) we see that the coupling capacitance C_{13} must vanish, since V_3 is arbitrary. This means that a change in V_3 will not affect Q_1, and vice versa. We then have electrostatic shielding between conducting bodies 1 and 3. Obviously, the same shielding effectiveness is obtained if the grounded conducting shell 2 encloses body 3 instead of body 1.

3–11 Electrostatic Energy and Forces

In Section 3–5 we indicated that electric potential at a point in an electric field is the work required to bring a unit positive charge from infinity (at reference zero-potential) to that point. To bring a charge Q_2 (slowly, so that kinetic energy and radiation effects may be neglected) from infinity *against* the field of a charge Q_1 in free space to a distance R_{12}, the amount of work required is

$$W_2 = Q_2V_2 = Q_2\frac{Q_1}{4\pi\epsilon_0 R_{12}}. \tag{3–159}$$

Because electrostatic fields are conservative, W_2 is independent of the path followed by Q_2. Another form of Eq. (3–159) is

$$W_2 = Q_1\frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1V_1. \tag{3–160}$$

This work is stored in the assembly of the two charges as potential energy. Combining Eqs. (3–159) and (3–160), we can write

$$W_2 = \tfrac{1}{2}(Q_1V_1 + Q_2V_2). \tag{3–161}$$

Now suppose another charge Q_3 is brought from infinity to a point that is R_{13} from Q_1 and R_{23} from Q_2; an additional amount of work is required that equals

$$\Delta W = Q_3V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right). \tag{3–162}$$

The sum of ΔW in Eq. (3–162) and W_2 in Eq. (3–159) is the potential energy, W_3, stored in the assembly of the three charges Q_1, Q_2, and Q_3. That is,

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1Q_2}{R_{12}} + \frac{Q_1Q_3}{R_{13}} + \frac{Q_2Q_3}{R_{23}}\right). \tag{3–163}$$

We can rewrite W_3 in the following form:

$$\begin{aligned} W_3 &= \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right)\right. \\ &\quad \left. + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right] \\ &= \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3). \end{aligned} \tag{3–164}$$

In Eq. (3–164), V_1, the potential at the position of Q_1, is caused by charges Q_2 and Q_3; it is *different* from the V_1 in Eq. (3–160) in the two-charge case. Similarly, V_2 and V_3 are the potentials at Q_2 and Q_3, respectively, in the three-charge assembly.

Extending this procedure of bringing in additional charges, we arrive at the following general expression for the potential energy of a group of N discrete point charges at rest. (The purpose of the subscript e on W_e is to denote that the energy is of an electric nature.) We have

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},} \tag{3–165}$$

where V_k, the electric potential at Q_k, is caused by all the other charges and has the following expression:

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1 \\ (j\neq k)}}^{N} \frac{Q_j}{R_{jk}}. \tag{3–166}$$

Two remarks are in order here. First, W_e can be negative. For instance, W_2 in Eq. (3–159) will be negative if Q_1 and Q_2 are of opposite signs. In that case, work is done by the field (not against the field) established by Q_1 in moving Q_2 from infinity. Second, W_e in Eq. (3–165) represents only the interaction energy (mutual energy) and does not include the work required to assemble the individual point charges themselves (self-energy).

The SI unit for energy, ***joule*** (J), is too large a unit for work in physics of elementary particles, where energy is more conveniently measured in terms of a much smaller unit called ***electron-volt*** (eV). An electron-volt is the energy or work required to move an electron against a potential difference of one volt.

$$1 \quad \text{(eV)} = (1.60 \times 10^{-19}) \times 1 = 1.60 \times 10^{-19} \qquad \text{(J)}. \tag{3–167}$$

Energy in (eV) is essentially that in (J) per unit electronic charge. The proton beams of the world's most powerful high-energy particle accelerator collide with a kinetic energy of two trillion electron-volts (2 TeV), or $(2 \times 10^{12}) \times (1.60 \times 10^{-19}) = 3.20 \times 10^{-7}$ (J). A binding energy of $W = 5 \times 10^{-19}$ (J) in an ionic crystal is equal to $W/e = 5 \times 10^{-19}/1.60 \times 10^{-19} = 3.125$ (eV), which is a more convenient number to use than the one in terms of joules.

EXAMPLE 3–22 Find the energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ.

Solution Because of symmetry, it is simplest to assume that the sphere of charge is assembled by bringing up a succession of spherical layers of thickness dR. At a radius R shown in Fig. 3–38 the potential is

$$V_R = \frac{Q_R}{4\pi\epsilon_0 R},$$

where Q_R is the total charge contained in a sphere of radius R:

$$Q_R = \rho \tfrac{4}{3}\pi R^3.$$

The differential charge in a spherical layer of thickness dR is

$$dQ_R = \rho 4\pi R^2\, dR,$$

and the work or energy in bringing up dQ_R is

$$dW = V_R\, dQ_R = \frac{4\pi}{3\epsilon_0}\rho^2 R^4\, dR.$$

Hence the total work or energy required to assemble a uniform sphere of charge of radius b and charge density ρ is

$$W = \int dW = \frac{4\pi}{3\epsilon_0}\rho^2 \int_0^b R^4\, dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0} \quad \text{(J)}. \tag{3–168}$$

In terms of the total charge

$$Q = \rho \frac{4\pi}{3} b^3,$$

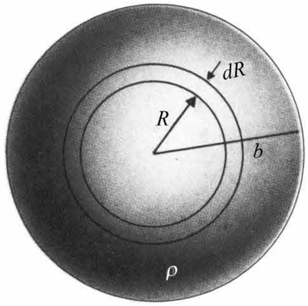


FIGURE 3–38
Assembling a uniform sphere of charge (Example 3–22).

we have

$$W = \frac{3Q^2}{20\pi\epsilon_0 b} \qquad \text{(J)}. \tag{3-169}$$

Equation (3–169) shows that the energy is directly proportional to the square of the total charge and inversely proportional to the radius. The sphere of charge in Fig. 3–38 could be a cloud of electrons, for instance. ■

For a continuous charge distribution of density ρ the formula for W_e in Eq. (3–165) for discrete charges must be modified. Without going through a separate proof we replace Q_k by $\rho\, dv$ and the summation by an integration and obtain

$$\boxed{W_e = \tfrac{1}{2}\int_{V'} \rho V\, dv \qquad \text{(J)}.} \tag{3-170}$$

In Eq. (3–170), V is the potential at the point where the volume charge density is ρ, and V' is the volume of the region where ρ exists.

EXAMPLE 3–23 Solve the problem in Example 3–22 by using Eq. (3–170).

Solution In Example 3–22 we solved the problem of assembling a sphere of charge by bringing up a succession of spherical layers of a differential thickness. Now we assume that the sphere of charge is already in place. Since ρ is a constant, it can be taken out of the integral sign. For a spherically symmetrical problem,

$$W_e = \frac{\rho}{2}\int_{V'} V\, dv = \frac{\rho}{2}\int_0^b V\, 4\pi R^2\, dR, \tag{3-171}$$

where V is the potential at a point R from the center. To find V at R, we must find the negative of the line integral of $\mathbf{E}$ in two regions: (1) $\mathbf{E}_1 = \mathbf{a}_R E_{R1}$ from $R = \infty$ to $R = b$, and (2) $\mathbf{E}_2 = \mathbf{a}_R E_{R2}$ from $R = b$ to $R = R$. We have

$$\mathbf{E}_{R1} = \mathbf{a}_R \frac{Q}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho b^3}{3\epsilon_0 R^2}, \qquad R \geq b,$$

and

$$\mathbf{E}_{R2} = \mathbf{a}_R \frac{Q_R}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho R}{3\epsilon_0}, \qquad 0 < R \leq b.$$

Consequently, we obtain

$$\begin{aligned} V &= -\int_\infty^R \mathbf{E}\cdot d\mathbf{R} = -\left[\int_\infty^b E_{R1}\, dR + \int_b^R E_{R2}\, dR\right] \\ &= -\left[\int_\infty^b \frac{\rho b^3}{3\epsilon_0 R^2}\, dR + \int_b^R \frac{\rho R}{3\epsilon_0}\, dR\right] \\ &= \frac{\rho}{3\epsilon_0}\left(b^2 + \frac{b^2}{2} - \frac{R^2}{2}\right) = \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right). \end{aligned} \tag{3-172}$$

Substituting Eq. (3–172) in Eq. (3–171), we get

$$W_e = \frac{\rho}{2}\int_0^b \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right)4\pi R^2\, dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0},$$

which is the same as the result in Eq. (3–168). ▬

Note that W_e in Eq. (3–170) includes the work (self-energy) required to assemble the distribution of macroscopic charges, because it is the energy of interaction of every infinitesimal charge element with all other infinitesimal charge elements. As a matter of fact, we have used Eq. (3–170) in Example 3–23 to find the self-energy of a uniform spherical charge. As the radius b approaches zero, the self-energy of a (mathematical) point charge of a given Q is infinite (see Eq. 3–169). The self-energies of point charges Q_k are not included in Eq. (3–165). Of course, there are, strictly, no point charges, inasmuch as the smallest charge unit, the electron, is itself a distribution of charge.

3–11.1 ELECTROSTATIC ENERGY IN TERMS OF FIELD QUANTITIES

In Eq. (3–170) the expression of electrostatic energy of a charge distribution contains the source charge density ρ and the potential function V. We frequently find it more convenient to have an expression of W_e in terms of field quantities $\mathbf{E}$ and/or $\mathbf{D}$, without knowing ρ explicitly. To this end, we substitute $\nabla \cdot \mathbf{D}$ for ρ in Eq. (3–170):

$$W_e = \tfrac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D})V\, dv. \tag{3–173}$$

Now, using the vector identity (from Problem P.2–28),

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V, \tag{3–174}$$

we can write Eq. (3–173) as

$$\begin{aligned} W_e &= \tfrac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})\, dv - \tfrac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V\, dv \\ &= \tfrac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n\, ds + \tfrac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\, dv, \end{aligned} \tag{3–175}$$

where the divergence theorem has been used to change the first volume integral into a closed surface integral and $\mathbf{E}$ has been substituted for $-\nabla V$ in the second volume integral. Since V' can be any volume that includes all the charges, we may choose it to be a very large sphere with radius R. As we let $R \to \infty$, electric potential V and the magnitude of electric displacement D fall off at least as fast as $1/R$ and $1/R^2$, respectively.† The area of the bounding surface S' increases as R^2. Hence the surface integral in Eq. (3–175) decreases at least as fast as $1/R$ and will vanish as $R \to \infty$. We are then left with only the second integral on the right side of Eq. (3–175).

† For point charges $V \propto 1/R$ and $D \propto 1/R^2$; for dipoles $V \propto 1/R^2$ and $D \propto 1/R^3$.

$$W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}\,dv \qquad \text{(J).} \tag{3–176a}$$

Using the relation $\mathbf{D} = \epsilon\mathbf{E}$ for a linear medium, Eq. (3–176a) can be written in two other forms:

$$W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)} \tag{3–176b}$$

and

$$W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J).} \tag{3–176c}$$

We can always define an ***electrostatic energy density*** w_e mathematically, such that its volume integral equals the total electrostatic energy:

$$W_e = \int_{V'} w_e\,dv. \tag{3–177}$$

We can therefore write

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad \text{(J/m}^3\text{)} \tag{3–178a}$$

or

$$w_e = \tfrac{1}{2}\epsilon E^2 \qquad \text{(J/m}^3\text{)} \tag{3–178b}$$

or

$$w_e = \frac{D^2}{2\epsilon} \qquad \text{(J/m}^3\text{).} \tag{3–178c}$$

However, this definition of energy density is artificial because a physical justification has not been found to localize energy with an electric field; all we know is that the volume integrals in Eqs. (3–176a, b, c) give the correct total electrostatic energy.

EXAMPLE 3–24 In Fig. 3–39 a parallel-plate capacitor of area S and separation d is charged to a voltage V. The permittivity of the dielectric is ϵ. Find the stored electrostatic energy.

Solution With the d-c source (batteries) connected as shown, the upper and lower plates are charged positive and negative, respectively. If the fringing of the field at

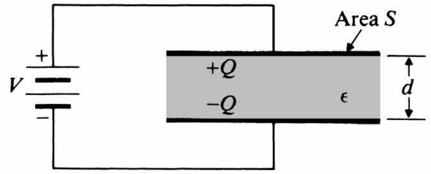


FIGURE 3–39
A charged parallel-plate capacitor (Example 3–24).

the edges is neglected, the electric field in the dielectric is uniform (over the plate) and constant (across the dielectric) and has a magnitude

$$E = \frac{V}{d}.$$

Using Eq. (3–176b), we have

$$W_e = \frac{1}{2}\int_{V'} \epsilon\left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2}\left(\epsilon\frac{S}{d}\right)V^2. \tag{3–179}$$

The quantity in the parentheses of the last expression, $\epsilon S/d$, is the capacitance of the parallel-plate capacitor (see Eq. 3–136). So,

$$\boxed{W_e = \tfrac{1}{2}CV^2 \qquad \text{(J)}.} \tag{3–180a}$$

Since $Q = CV$, Eq. (3–180a) can be put in two other forms:

$$\boxed{W_e = \tfrac{1}{2}QV \qquad \text{(J)}} \tag{3–180b}$$

and

$$\boxed{W_e = \frac{Q^2}{2C} \qquad \text{(J)}.} \tag{3–180c}$$

It so happens that Eqs. (3–180a, b, c) hold true for any two-conductor capacitor (see Problem P.3–43).

EXAMPLE 3–25 Use energy formulas (3–176) and (3–180) to find the capacitance of a cylindrical capacitor having a length L, an inner conductor of radius a, an outer conductor of inner radius b, and a dielectric of permittivity ϵ, as shown in Fig. 3–29.

Solution By applying Gauss's law, we know that

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}, \qquad a < r < b.$$

The electrostatic energy stored in the dielectric region is, from Eq. (3–176b),

$$\begin{aligned} W_e &= \frac{1}{2}\int_a^b \epsilon\left(\frac{Q}{2\pi\epsilon L r}\right)^2 (L2\pi r\,dr) \\ &= \frac{Q^2}{4\pi\epsilon L}\int_a^b \frac{dr}{r} = \frac{Q^2}{4\pi\epsilon L}\ln\frac{b}{a}. \end{aligned} \tag{3–181}$$

On the other hand, W_e can also be expressed in the form of Eq. (3–180c). Equating (3–180c) and (3–181), we obtain

$$\frac{Q^2}{2C} = \frac{Q^2}{4\pi\epsilon L} \ln \frac{b}{a}$$

or

$$C = \frac{2\pi\epsilon L}{\ln \frac{b}{a}},$$

which is the same as that given in Eq. (3–139).

3–11.2 ELECTROSTATIC FORCES

Coulomb's law governs the force between two point charges. In a more complex system of charged bodies, using Coulomb's law to determine the force on one of the bodies that is caused by the charges on other bodies would be very tedious. This would be so even in the simple case of finding the force between the plates of a charged parallel-plate capacitor. We will now discuss a method for calculating the force on an object in a charged system from the electrostatic energy of the system. This method is based on the ***principle of virtual displacement***. We will consider two cases: (1) that of an isolated system of bodies with fixed charges, and (2) that of a system of conducting bodies with fixed potentials.

System of Bodies with Fixed Charges We consider an isolated system of charged conducting, as well as dielectric, bodies separated from one another with no connection to the outside world. The charges on the bodies are constant. Imagine that the electric forces have displaced one of the bodies by a differential distance $d\ell$ (a virtual displacement). The mechanical work done *by the system* would be

$$dW = \mathbf{F}_Q \cdot d\boldsymbol{\ell}, \tag{3–182}$$

where $\mathbf{F}_Q$ is the total electric force acting on the body under the condition of constant charges. Since we have an isolated system with no external supply of energy, this mechanical work must be done at the expense of the stored electrostatic energy; that is,

$$dW = -dW_e = \mathbf{F}_Q \cdot d\boldsymbol{\ell}. \tag{3–183}$$

Noting from Eq. (2–88) in Section 2–6 that the differential change of a scalar resulting from a position change $d\boldsymbol{\ell}$ is the dot product of the gradient of the scalar, and $d\boldsymbol{\ell}$, we write

$$dW_e = (\nabla W_e) \cdot d\boldsymbol{\ell}. \tag{3–184}$$

Since $d\boldsymbol{\ell}$ is arbitrary, comparison of Eqs. (3–183) and (3–184) leads to

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}.} \tag{3–185}$$

Equation (3–185) is a very simple formula for the calculation of $\mathbf{F}_Q$ from the electrostatic energy of the system. In Cartesian coordinates the component forces are

$$(F_Q)_x = -\frac{\partial W_e}{\partial x}, \tag{3–186a}$$

$$(F_Q)_y = -\frac{\partial W_e}{\partial y}, \tag{3–186b}$$

$$(F_Q)_z = -\frac{\partial W_e}{\partial z}. \tag{3–186c}$$

If the body under consideration is constrained to rotate about an axis, say the z-axis, the mechanical work done by the system for a virtual angular displacement $d\phi$ would be

$$dW = (T_Q)_z \, d\phi, \tag{3–187}$$

where $(T_Q)_z$ is the z-component of the torque acting on the body under the condition of constant charges. The foregoing procedure will lead to

$$\boxed{(T_Q)_z = -\frac{\partial W_e}{\partial \phi} \qquad (\text{N}\cdot\text{m}).} \tag{3–188}$$

System of Conducting Bodies with Fixed Potentials Now consider a system in which conducting bodies are held at fixed potentials through connections to such external sources as batteries. Uncharged dielectric bodies may also be present. A displacement $d\ell$ by a conducting body would result in a change in total electrostatic energy and would require the sources to transfer charges to the conductors in order to keep them at their fixed potentials. If a charge dQ_k (which may be positive or negative) is added to the kth conductor that is maintained at potential V_k, the work done or energy supplied by the sources is $V_k \, dQ_k$. The total energy supplied by the sources to the system is

$$dW_s = \sum_k V_k \, dQ_k. \tag{3–189}$$

The mechanical work done by the system as a consequence of the virtual displacement is

$$dW = \mathbf{F}_V \cdot d\boldsymbol{\ell}, \tag{3–190}$$

where $\mathbf{F}_V$ is the electric force on the conducting body under the condition of constant potentials. The charge transfers also change the electrostatic energy of the system by an amount dW_e, which, in view of Eq. (3–165), is

$$dW_e = \frac{1}{2}\sum_k V_k \, dQ_k = \frac{1}{2} \, dW_s. \tag{3–191}$$

Conservation of energy demands that

$$dW + dW_e = dW_s. \tag{3-192}$$

Substitution of Eqs. (3–189), (3–190), and (3–191) in Eq. (3–192) gives

$$\mathbf{F}_V \cdot d\boldsymbol{\ell} = dW_e$$
$$= (\nabla W_e) \cdot d\boldsymbol{\ell}$$

or

$$\boxed{\mathbf{F}_V = \nabla W_e \qquad \text{(N)}.} \tag{3-193}$$

Comparison of Eqs. (3–193) and (3–185) reveals that the only difference between the formulas for the electric forces in the two cases is in the sign. It is clear that if the conducting body is constrained to rotate about the z-axis, the z-component of the electric torque will be

$$\boxed{(T_V)_z = \frac{\partial W_e}{\partial \phi} \qquad \text{(N·m)},} \tag{3-194}$$

which differs from Eq. (3–188) also only by a sign change.

EXAMPLE 3–26 Determine the force on the conducting plates of a charged parallel-plate capacitor. The plates have an area S and are separated in air by a distance x.

Soiution We solve the problem in two ways: (a) by assuming fixed charges, and then (b) by assuming fixed potentials. The fringing of field around the edges of the plates will be neglected.

a) *Fixed charges.* With fixed charges $\pm Q$ on the plates, an electric field intensity $E_x = Q/(\epsilon_0 S) = V/x$ exists in the air between the plates regardless of their separation (unchanged by a virtual displacement). From Eq. (3–180b),

$$W_e = \tfrac{1}{2}QV = \tfrac{1}{2}QE_x x,$$

where Q and E_x are constants. Using Eq. (3–186a), we obtain

$$(F_Q)_x = -\frac{\partial}{\partial x}\left(\frac{1}{2}QE_x x\right) = -\frac{1}{2}QE_x = -\frac{Q^2}{2\epsilon_0 S}, \tag{3-195}$$

where the negative signs indicate that the force is opposite to the direction of increasing x. It is an attractive force.

b) *Fixed potentials.* With fixed potentials it is more convenient to use the expression in Eq. (3–180a) for W_e. Capacitance C for the parallel-plate air capacitor is $\epsilon_0 S/x$. We have, from Eq. (3–193),

$$(F_V)_x = \frac{\partial W_e}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial}{\partial x}\left(\frac{\epsilon_0 S}{x}\right) = -\frac{\epsilon_0 S V^2}{2x^2}. \tag{3-196}$$

How different are $(F_Q)_x$ in Eq. (3–195) and $(F_V)_x$ in Eq. (3–196)? Recalling the relation

$$Q = CV = \frac{\epsilon_0 SV}{x},$$

we find

$$(F_Q)_x = (F_V)_x. \tag{3–197}$$

The force is the same in both cases in spite of the apparent sign difference in the formulas as expressed by Eqs. (3–185) and (3–193). A little reflection on the physical problem will convince us that this must be true. Since the charged capacitor has fixed dimensions, a given Q will result in a fixed V, and vice versa. Therefore there is a unique force between the plates regardless of whether Q or V is given, and the force certainly does not depend on virtual displacements. A change in the conceptual constraint (fixed Q or fixed V) cannot change the unique force between the plates.

The preceding discussion holds true for a general charged two-conductor capacitor with capacitance C. The electrostatic force F_ℓ in the direction of a virtual displacement $d\ell$ for fixed charges is

$$(F_Q)_\ell = -\frac{\partial W_e}{\partial \ell} = -\frac{\partial}{\partial \ell}\left(\frac{Q^2}{2C}\right) = \frac{Q^2}{2C^2}\frac{\partial C}{\partial \ell}. \tag{3–198}$$

For fixed potentials,

$$(F_V)_\ell = \frac{\partial W_e}{\partial \ell} = \frac{\partial}{\partial \ell}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial C}{\partial \ell} = \frac{Q^2}{2C^2}\frac{\partial C}{\partial \ell}. \tag{3–199}$$

It is clear that the forces calculated from the two procedures, which assumed different constraints imposed on the same charged capacitor, are equal.

Review Questions

R.3–1 Write the differential form of the fundamental postulates of electrostatics in free space.

R.3–2 Under what conditions will the electric field intensity be both solenoidal and irrotational?

R.3–3 Write the integral form of the fundamental postulates of electrostatics in free space, and state their meaning in words.

R.3–4 When the formula for the electric field intensity of a point charge, Eq. (3–12), was derived,

a) why was it necessary to stipulate that q is in a boundless free space?
b) why did we *not* construct a cubic or a cylindrical surface around q?

R.3–5 In what ways does the electric field intensity vary with distance for

a) a point charge? **b)** an electric dipole?

R.3–6 State *Coulomb's law.*

R.3–7 Explain the principle of operation of ink-jet printers.

R.3–8 State *Gauss's law.* Under what conditions is Gauss's law especially useful in determining the electric field intensity of a charge distribution?

R.3–9 Describe the ways in which the electric field intensity of an infinitely long, straight line charge of uniform density varies with distance.

R.3–10 Is Gauss's law useful in finding the **E** field of a finite line charge? Explain.

R.3–11 See Example 3–6, Fig. 3–9. Could a cylindrical pillbox with circular top and bottom faces be chosen as a Gaussian surface? Explain.

R.3–12 Make a two-dimensional sketch of the electric field lines and the equipotential lines of a point charge.

R.3–13 At what value of θ is the **E** field of a z-directed electric dipole pointed in the negative z-direction?

R.3–14 Refer to Eq. (3–64). Explain why the absolute sign around z is required.

R.3–15 If the electric potential at a point is zero, does it follow that the electrical field intensity is also zero at that point? Explain.

R.3–16 If the electric field intensity at a point is zero, does it follow that the electric potential is also zero at that point? Explain.

R.3–17 If an uncharged spherical conducting shell of a finite thickness is placed in an external electric field $\mathbf{E}_o$, what is the electric field intensity at the center of the shell? Describe the charge distributions on both the outer and the inner surfaces of the shell.

R.3–18 What are *electrets*? How can they be made?

R.3–19 Can $\nabla'(1/R)$ in Eq. (3–84) be replaced by $\nabla(1/R)$? Explain.

R.3–20 Define *polarization vector.* What is its SI unit?

R.3–21 What are *polarization charge densities*? What are the SI units for $\mathbf{P} \cdot \mathbf{a}_n$ and $\nabla \cdot \mathbf{P}$?

R.3–22 What do we mean by *simple medium*?

R.3–23 What properties do *anisotropic materials* have?

R.3–24 What characterizes a *uniaxial medium*?

R.3–25 Define *electric displacement vector.* What is its SI unit?

R.3–26 Define *electric susceptibility.* What is its unit?

R.3–27 What is the difference between the *permittivity* and the *dielectric constant* of a medium?

R.3–28 Does the electric flux density due to a given charge distribution depend on the properties of the medium? Does the electric field intensity? Explain.

R.3–29 What is the difference between the *dielectric constant* and the *dielectric strength* of a dielectric material?

R.3–30 Explain the principle of operation of lightning arresters.

R.3–31 What are the general boundary conditions for electrostatic fields at an interface between two different dielectric media?

R.3–32 What are the boundary conditions for electrostatic fields at an interface between a conductor and a dielectric with permittivity ϵ?

R.3–33 What is the boundary condition for electrostatic potential at an interface between two different dielectric media?

R.3–34 Does a force exist between a point charge and a dielectric body? Explain.

R.3–35 Define *capacitance* and *capacitor*.

R.3–36 Assume that the permittivity of the dielectric in a parallel-plate capacitor is not constant. Will Eq. (3–136) hold if the average value of permittivity is used for ϵ in the formula? Explain.

R.3–37 Given three 1-μF capacitors, explain how they should be connected in order to obtain a total capacitance of

a) $\frac{1}{3}$ (μF), **b)** $\frac{2}{3}$ (μF), **c)** $\frac{3}{2}$ (μF), **d)** 3 (μF).

R.3–38 What are *coefficients of potential*, *coefficients of capacitance*, and *coefficients of induction*?

R.3–39 What are *partial capacitances*? How are they different from coefficients of capacitance?

R.3–40 Explain the principle of electrostatic shielding.

R.3–41 What is the definition of an *electron-volt*? How does it compare with a joule?

R.3–42 What is the expression for the electrostatic energy of an assembly of four discrete point charges?

R.3–43 What is the expression for the electrostatic energy of a continuous distribution of charge in a volume? on a surface? along a line?

R.3–44 Provide a mathematical expression for electrostatic energy in terms of **E** and/or **D**.

R.3–45 Discuss the meaning and use of the *principle of virtual displacement*.

R.3–46 What is the relation between the force and the stored energy in a system of stationary charged objects under the condition of constant charges? Under the condition of fixed potentials?

Problems

P.3–1 Refer to Fig. 3–4.

a) Find the relation between the angle of arrival, α, of the electron beam at the screen and the deflecting electric field intensity E_d.

b) Find the relation between w and L such that $d_1 = d_0/20$.

P.3–2 The cathode-ray oscilloscope (CRO) shown in Fig. 3–4 is used to measure the voltage applied to the parallel deflection plates.

a) Assuming no breakdown in insulation, what is the maximum voltage that can be measured if the distance of separation between the plates is h?

b) What is the restriction on L if the diameter of the screen is D?

c) What can be done with a fixed geometry to double the CRO's maximum measurable voltage?

P.3–3 The deflection system of a cathode-ray oscilloscope usually consists of two pairs of parallel plates producing orthogonal electric fields. Assume the presence of another set of plates in Fig. 3–4 that establishes a uniform electric field $\mathbf{E}_x = \mathbf{a}_x E_x$ in the deflection region. Deflection voltages $v_x(t)$ and $v_y(t)$ are applied to produce $\mathbf{E}_x$ and $\mathbf{E}_y$, respectively. Determine the types of waveforms that $v_x(t)$ and $v_y(t)$ should have if the electrons are to trace the following graphs on the fluorescent screen:

a) a horizontal line,
b) a straight line having a negative unity slope,
c) a circle,
d) two cycles of a sine wave.

P.3–4 Write a short article explaining the principle of operation of xerography. (Use library resources if needed.)

P.3–5 Two point charges, Q_1 and Q_2, are located at (1, 2, 0) and (2, 0, 0), respectively. Find the relation between Q_1 and Q_2 such that the total force on a test charge at the point $P(-1, 1, 0)$ will have

a) no x-component, **b)** no y-component.

P.3–6 Two very small conducting spheres, each of a mass 1.0×10^{-4} (kg), are suspended at a common point by very thin nonconducting threads of a length 0.2 (m). A charge Q is placed on each sphere. The electric force of repulsion separates the spheres, and an equilibrium is reached when the suspending threads make an angle of 10°. Assuming a gravitational force of 9.80 (N/kg) and a negligible mass for the threads, find Q.

P.3–7 Find the force between a charged circular loop of radius b and uniform charge density ρ_ℓ and a point charge Q located on the loop axis at a distance h from the plane of the loop. What is the force when $h \gg b$, and when $h = 0$? Plot the force as a function of h.

P.3–8 A line charge of uniform density ρ_ℓ in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

P.3–9 Three uniform line charges—$\rho_{\ell 1}$, $\rho_{\ell 2}$, and $\rho_{\ell 3}$, each of length L—form an equilateral triangle. Assuming that $\rho_{\ell 1} = 2\rho_{\ell 2} = 2\rho_{\ell 3}$, determine the electric field intensity at the center of the triangle.

P.3–10 Assuming that the electric field intensity is $\mathbf{E} = \mathbf{a}_x 100x$ (V/m), find the total electric charge contained inside

a) a cubical volume 100 (mm) on a side centered symmetrically at the origin,
b) a cylindrical volume around the z-axis having a radius 50 (mm) and a height 100 (mm) centered at the origin.

P.3–11 A spherical distribution of charge $\rho = \rho_0[1 - (R^2/b^2)]$ exists in the region $0 \le R \le b$. This charge distribution is concentrically surrounded by a conducting shell with inner radius R_i $(>b)$ and outer radius R_o. Determine $\mathbf{E}$ everywhere.

P.3–12 Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ $(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb}, respectively.

a) Determine $\mathbf{E}$ everywhere.
b) What must be the relation between a and b in order that $\mathbf{E}$ vanishes for $r > b$?

P.3–13 Determine the work done in carrying a -2 (μC) charge from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$ in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$

a) along the parabola $x = 2y^2$,
b) along the straight line joining P_1 and P_2.

P.3–14 At what values of θ does the electric field intensity of a z-directed dipole have no z-component?

P.3–15 Three charges $(+q, -2q$, and $+q)$ are arranged along the z-axis at $z = d/2$, $z = 0$, and $z = -d/2$, respectively.

a) Determine V and $\mathbf{E}$ at a distant point $P(R, \theta, \phi)$.
b) Find the equations for equipotential surfaces and streamlines.
c) Sketch a family of equipotential lines and streamlines.
(Such an arrangement of three charges is called a ***linear electrostatic quadrupole***.)

P.3–16 A finite line charge of length L carrying uniform line charge density ρ_ℓ is coincident with the x-axis.
a) Determine V in the plane bisecting the line charge.
b) Determine $\mathbf{E}$ from ρ_ℓ directly by applying Coulomb's law.
c) Check the answer in part (b) with $-\nabla V$.

P.3–17 In Example 3–5 we obtained the electric field intensity around an infinitely long line charge of a uniform charge density in a very simple manner by applying Gauss's law. Since $|\mathbf{E}|$ is a function of r only, any coaxial cylinder around the infinite line charge is an equipotential surface. In practice, all conductors are of finite length. A finite line charge carrying a constant charge density ρ_ℓ along the axis, however, does not produce a constant potential on a concentric cylindrical surface. Given the finite line charge ρ_ℓ of length L in Fig. 3–40, find the potential on the cylindrical surface of radius b as a function of x and plot it.

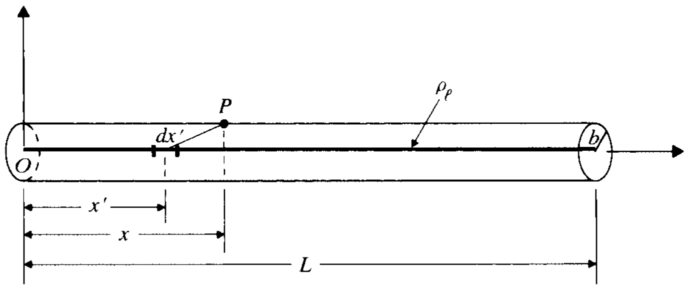


FIGURE 3–40
A finite line charge (Problem P.3–17).

(*Hint:* Find dV at P due to charge $\rho_\ell dx'$ and integrate.)

P.3–18 A charge Q is distributed uniformly over an $L \times L$ square plate. Determine V and $\mathbf{E}$ at a point on the axis perpendicular to the plate and through its center.

P.3–19 A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\mathbf{E}$ on its axis
a) at a point outside the tube, then
b) at a point inside the tube.

P.3–20 An early model of the atomic structure of a chemical element was that the atom was a spherical cloud of uniformly distributed positive charge Ne, where N is the atomic number and e is the magnitude of electronic charge. Electrons, each carrying a negative charge $-e$, were considered to be imbedded in the cloud. Assuming the spherical charge cloud to have a radius R_0 and neglecting collision effects,
a) find the force experienced by an imbedded electron at a distance r from the center;
b) describe the motion of the electron;
c) explain why this atomic model is unsatisfactory.

P.3–21 A simple classical model of an atom consists of a nucleus of a positive charge Ne surrounded by a spherical electron cloud of the same total negative charge. (N is the atomic number and e is the magnitude of electronic charge.) An external electric field $\mathbf{E}_o$ will cause the nucleus to be displaced a distance r_o from the center of the electron cloud, thus polarizing the atom. Assuming a uniform charge distribution within the electron cloud of radius b, find r_o.

P.3–22 The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_o(\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z)$.

a) Determine the surface and volume bound-charge densities.
b) Show that the total bound charge is zero.

P.3–23 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization $\mathbf{P}$ exists.

P.3–24 Solve the following problems:

a) Find the breakdown voltage of a parallel-plate capacitor, assuming that conducting plates are 50 (mm) apart and the medium between them is air.
b) Find the breakdown voltage if the entire space between the conducting plates is filled with plexiglass, which has a dielectric constant 3 and a dielectric strength 20 (kV/mm).
c) If a 10-(mm) thick plexiglass is inserted between the plates, what is the maximum voltage that can be applied to the plates without a breakdown?

P.3–25 Assume that the $z = 0$ plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\mathbf{E}_1$ in region 1 is $\mathbf{a}_x 2y - \mathbf{a}_y 3x + \mathbf{a}_z(5 + z)$, what do we also know about $\mathbf{E}_2$ and $\mathbf{D}_2$ in region 2? Can we determine $\mathbf{E}_2$ and $\mathbf{D}_2$ at any point in region 2? Explain.

P.3–26 Determine the boundary conditions for the tangential and the normal components of $\mathbf{P}$ at an interface between two perfect dielectric media with dielectric constants ϵ_{r1} and ϵ_{r2}.

P.3–27 What are the boundary conditions that must be satisfied by the electric potential at an interface between two perfect dielectrics with dielectric constants ϵ_{r1} and ϵ_{r2}?

P.3–28 Dielectric lenses can be used to collimate electromagnetic fields. In Fig. 3–41 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\mathbf{E}_1$ at point $P(r_o, 45°, z)$ in region 1 is $\mathbf{a}_r 5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that $\mathbf{E}_3$ in region 3 is parallel to the x-axis?

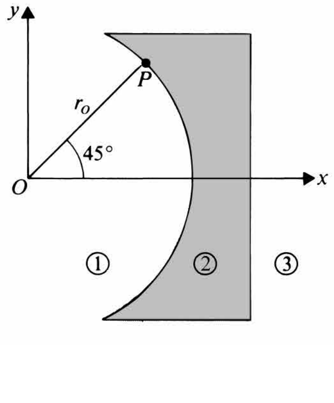


FIGURE 3–41
A dielectric lens (Problem P.3–28).

P.3–29 Refer to Example 3–16. Assuming the same r_i and r_o and requiring the maximum electric field intensities in the insulating materials not to exceed 25% of their dielectric strengths, determine the voltage rating of the coaxial cable

a) if $r_p = 1.75r_i$;

b) if $r_p = 1.35r_i$.

c) Plot the variations of E_r and V versus r for both part (a) and part (b).

P.3–30 The space between a parallel-plate capacitor of area S is filled with a dielectric whose permittivity varies linearly from ϵ_1 at one plate ($y = 0$) to ϵ_2 at the other plate ($y = d$). Neglecting fringing effect, find the capacitance.

P.3–31 Assume that the outer conductor of the cylindrical capacitor in Example 3–18 is grounded and that the inner conductor is maintained at a potential V_0.

a) Find the electric field intensity, $\mathbf{E}(a)$, at the surface of the inner conductor.

b) With the inner radius, b, of the outer conductor fixed, find a so that $E(a)$ is minimized.

c) Find this minimum $E(a)$.

d) Determine the capacitance under the conditions of part (b).

P.3–32 The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

P.3–33 A cylindrical capacitor of length L consists of coaxial conducting surfaces of radii r_i and r_o. Two dielectric media of different dielectric constants ϵ_{r1} and ϵ_{r2} fill the space between the conducting surfaces as shown in Fig. 3–42. Determine its capacitance.

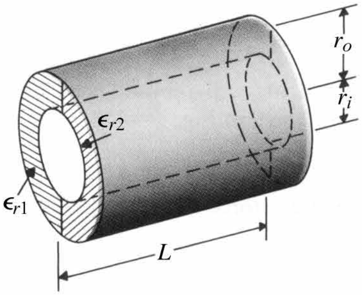


FIGURE 3–42
A cylindrical capacitor with two dielectric media (Problem P.3–33).

P.3–34 A capacitor consists of two coaxial metallic cylindrical surfaces of a length 30 (mm) and radii 5 (mm) and 7 (mm). The dielectric material between the surfaces has a relative permittivity $\epsilon_r = 2 + (4/r)$, where r is measured in mm. Determine the capacitance of the capacitor.

P.3–35 Assuming the earth to be a large conducting sphere (radius = 6.37×10^3 km) surrounded by air, find

a) the capacitance of the earth;

b) the maximum charge that can exist on the earth before the air breaks down.

P.3–36 Determine the capacitance of an isolated conducting sphere of radius b that is coated with a dielectric layer of uniform thickness d. The dielectric has an electric susceptibility χ_e.

P.3–37 A capacitor consists of two concentric spherical shells of radii R_i and R_o. The space between them is filled with a dielectric of relative permittivity ϵ_r from R_i to $b(R_i < b < R_o)$ and another dielectric of relative permittivity $2\epsilon_r$ from b to R_o.

a) Determine **E** and **D** everywhere in terms of an applied voltage V.

b) Determine the capacitance.

P.3–38 The two parallel conducting wires of a power transmission line have a radius a and are spaced at a distance d apart. The wires are at a height h above the ground. Assuming the ground to be perfectly conducting and both d and h to be much larger than a, find the expressions for the mutual and self-partial capacitances per unit length.

P.3–39 An isolated system consists of three very long parallel conducting wires. The axes of all three wires lie in a plane. The two outside wires are of a radius b and both are at a distance $d = 500b$ from a center wire of a radius $2b$. Determine the partial capacitances per unit length.

P.3–40 Calculate the amount of electrostatic energy of a uniform sphere of charge with radius b and volume charge density ρ stored in the following regions:

a) inside the sphere,

b) outside the sphere.

Check your results with those in Example 3–22.

P.3–41 Einstein's theory of relativity stipulates that the work required to assemble a charge is stored as energy in the mass and is equal to mc^2, where m is the mass and $c \cong 3 \times 10^8$ (m/s) is the velocity of light. Assuming the electron to be a perfect sphere, find its radius from its charge and mass (9.1×10^{-31} kg).

P.3–42 Find the electrostatic energy stored in the region of space $R > b$ around an electric dipole of moment **p**.

P.3–43 Prove that Eqs. (3–180) for stored electrostatic energy hold true for any two-conductor capacitor.

P.3–44 A parallel-plate capacitor of width w, length L, and separation d is partially filled with a dielectric medium of dielectric constant ϵ_r, as shown in Fig. 3–43. A battery of V_0 volts is connected between the plates.

a) Find **D**, **E**, and ρ_s in each region.

b) Find distance x such that the electrostatic energy stored in each region is the same.

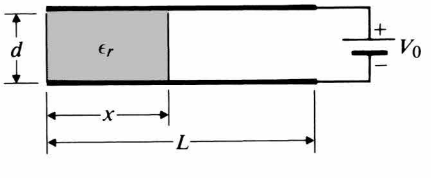


FIGURE 3–43
A parallel-plate capacitor (Problem P.3–44).

P.3–45 Using the principle of virtual displacement, derive an expression for the force between two point charges $+Q$ and $-Q$ separated by a distance x in free space.

P.3–46 A constant voltage V_0 is applied to a partially filled parallel-plate capacitor shown in Fig. 3–44. The permittivity of the dielectric is ϵ, and the area of the plates is S. Find the force on the upper plate.

P.3–47 The conductors of an isolated two-wire transmission line, each of radius b, are spaced at a distance D apart. Assuming $D \gg b$ and a voltage V_0 between the lines, find the force per unit length on the lines.

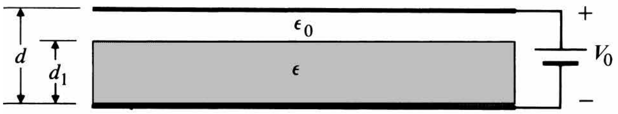


FIGURE 3–44
A parallel-plate capacitor (Problem P.3–46).

P.3–48 A parallel-plate capacitor of width w, length L, and separation d has a solid dielectric slab of permittivity ϵ in the space between the plates. The capacitor is charged to a voltage V_0 by a battery, as indicated in Fig. 3–45. Assuming that the dielectric slab is withdrawn to the position shown, determine the force acting on the slab

a) with the switch closed,
b) after the switch is first opened.

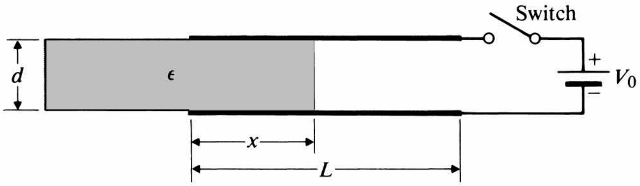


FIGURE 3–45
A partially filled parallel-plate capacitor (Problem P.3–48).