5

Steady Electric Currents

5–1 Introduction

In Chapters 3 and 4 we dealt with electrostatic problems, field problems associated with electric charges at rest. We now consider the charges in motion that constitute current flow. There are several types of electric currents caused by the *motion of free charges.*† ***Conduction currents*** in conductors and semiconductors are caused by drift motion of conduction electrons and/or holes; ***electrolytic currents*** are the result of migration of positive and negative ions; and ***convection currents*** result from motion of electrons and/or ions in a vacuum. In this chapter we shall pay special attention to conduction currents that are governed by Ohm's law. We will proceed from the point form of Ohm's law that relates current density and electric field intensity and obtain the $V = IR$ relationship in circuit theory. We will also introduce the concept of electromotive force and derive the familiar Kirchhoff's voltage law. Using the principle of ***conservation of charge***, we will show how to obtain a point relationship between current and charge densities, a relationship called the ***equation of continuity*** from which Kirchhoff's current law follows.

When a current flows across the interface between two media of different conductivities, certain boundary conditions must be satisfied, and the direction of current flow is changed. We will discuss these boundary conditions. We will also show that for a homogeneous conducting medium, the current density can be expressed as the gradient of a scalar field, which satisfies Laplace's equation. Hence, an analogous situation exists between steady-current and electrostatic fields that is the basis for mapping the potential distribution of an electrostatic problem in an ***electrolytic tank***.

The electrolyte in an electrolytic tank is essentially a liquid medium with a low conductivity, usually a diluted salt solution. Highly conducting metallic electrodes

† In a time-varying situation there is another type of current caused by bound charges. The time-rate of change of electric displacement leads to a ***displacement current***. This will be discussed in Chapter 7.

are inserted in the solution. When a voltage or potential difference is applied to the electrodes, an electric field is established within the solution, and the molecules of the electrolyte are decomposed into oppositely charged ions by a chemical process called ***electrolysis***. Positive ions move in the direction of the electric field, and negative ions move in a direction opposite to the field, both contributing to a current flow in the direction of the field. An experimental model can be set up in an electrolytic tank, with electrodes of proper geometrical shapes simulating the boundaries in electrostatic problems. The measured potential distribution in the electrolyte is then the solution to Laplace's equation for difficult-to-solve analytic problems having complex boundaries in a homogeneous medium.

Convection currents are the result of the motion of positively or negatively charged particles in a vacuum or rarefied gas. Familiar examples are electron beams in a cathode-ray tube and the violent motions of charged particles in a thunderstorm. Convection currents, the result of hydrodynamic motion involving a mass transport, are not governed by Ohm's law.

The mechanism of conduction currents is different from that of both electrolytic currents and convection currents. In their normal state the atoms of a conductor occupy regular positions in a crystalline structure. The atoms consist of positively charged nuclei surrounded by electrons in a shell-like arrangement. The electrons in the inner shells are tightly bound to the nuclei and are not free to move away. The electrons in the outermost shells of a conductor atom do not completely fill the shells; they are valence or conduction electrons and are only very loosely bound to the nuclei. These latter electrons may wander from one atom to another in a random manner. The atoms, on the average, remain electrically neutral, and there is no net drift motion of electrons. When an external electric field is applied on a conductor, an organized motion of the conduction electrons will result, producing an electric current. The average drift velocity of the electrons is very low (on the order of 10^{-4} or 10^{-3} m/s) even for very good conductors because they collide with the atoms in the course of their motion, dissipating part of their kinetic energy as heat. Even with the drift motion of conduction electrons, a conductor remains electrically neutral. Electric forces prevent excess electrons from accumulating at any point in a conductor. We will show analytically that the charge density in a conductor decreases exponentially with time. In a good conductor the charge density diminishes extremely rapidly toward zero as the state of equilibrium is approached.

5–2 Current Density and Ohm's Law

Consider the steady motion of one kind of charge carriers, each of charge q (which is negative for electrons), across an element of surface Δs with a velocity $\mathbf{u}$, as shown in Fig. 5–1. If N is the number of charge carriers per unit volume, then in time Δt each charge carrier moves a distance $\mathbf{u}\,\Delta t$, and the amount of charge passing through the surface Δs is

$$\Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \,\Delta s\,\Delta t \qquad \text{(C).} \tag{5–1}$$

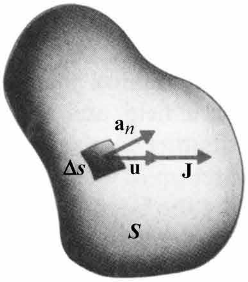


FIGURE 5–1
Conduction current due to drift motion of charge carriers across a surface.

Since current is the time rate of change of charge, we have

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta\mathbf{s} \qquad \text{(A)}. \tag{5–2}$$

In Eq. (5–2) we have written $\Delta\mathbf{s} = \mathbf{a}_n \Delta s$ as a vector quantity. It is convenient to define a vector point function, ***volume current density***, or simply ***current density***, **J**, in amperes per *square* meter,

$$\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2), \tag{5–3}$$

so that Eq. (5–2) can be written as

$$\Delta I = \mathbf{J} \cdot \Delta\mathbf{s}. \tag{5–4}$$

The total current I flowing through an arbitrary surface S is then the flux of the **J** vector through S:

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.} \tag{5–5}$$

Noting that the product Nq is in fact free charge per unit volume, we may rewrite Eq. (5–3) as

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),} \tag{5–6}$$

which is the relation between the ***convection current density*** and the velocity of the charge carrier.

EXAMPLE 5–1 In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

Solution The region between the cathode and the anode is shown in Fig. 5–2, where a cloud of electrons (negative space charge) exists such that the force of repulsion makes the electrons boiled off the hot cathode leave essentially with a zero velocity. In other words, the net electric field at the cathode is zero. Neglecting fringing effects, we have

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left. \frac{dV(y)}{dy} \right|_{y=0} = 0. \tag{5–7}$$

In the steady state the current density is constant, independent of y:

$$\mathbf{J} = -\mathbf{a}_y J = \mathbf{a}_y \rho(y) u(y), \tag{5–8}$$

where the charge density $\rho(y)$ is a negative quantity. The velocity $\mathbf{u} = \mathbf{a}_y u(y)$ is related to the electric field intensity $\mathbf{E}(y) = \mathbf{a}_y E(y)$ by Newton's law of motion:

$$m \frac{du(y)}{dt} = -eE(y) = e \frac{dV(y)}{dy}, \tag{5–9}$$

where $m = 9.11 \times 10^{-31}$ (kg) and $-e = -1.60 \times 10^{-19}$ (C) are the mass and charge, respectively, of an electron. Noting that

$$m \frac{du}{dt} = m \frac{du}{dy} \frac{dy}{dt} = mu \frac{du}{dy} = \frac{d}{dy}\left(\frac{1}{2} mu^2\right),$$

we can rewrite Eq. (5–9) as

$$\frac{d}{dy}\left(\frac{1}{2} mu^2\right) = e \frac{dV}{dy}. \tag{5–10}$$

Integration of Eq. (5–10) gives

$$\tfrac{1}{2} mu^2 = eV, \tag{5–11}$$

where the constant of integration has been set to zero because at $y = 0$, $u(0) = V(0) = 0$. From Eq. (5–11) we obtain

$$u = \left(\frac{2e}{m} V\right)^{1/2}. \tag{5–12}$$

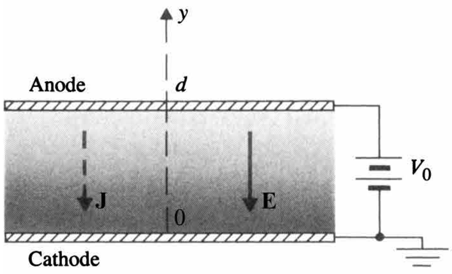


FIGURE 5–2
Space-charge-limited vacuum diode (Example 5–1).

In order to find $V(y)$ in the interelectrode region we must solve Poisson's equation with ρ expressed in terms of $V(y)$ from Eq. (5–8):

$$\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–13}$$

We have, from Eq. (4–6),

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–14}$$

Equation (5–14) can be integrated if both sides are first multiplied by $2\,dV/dy$. The result is

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{1/2} + c. \tag{5–15}$$

At $y = 0$, $V = 0$, and $dV/dy = 0$ from Eq. (5–7), so $c = 0$. Equation (5–15) becomes

$$V^{-1/4}\,dV = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy. \tag{5–16}$$

Integrating the left side of Eq. (5–16) from $V = 0$ to V_0 and the right side from $y = 0$ to d, we obtain

$$\frac{4}{3}\,V_0^{3/4} = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} d,$$

or

$$J = \frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}\,V_0^{3/2} \qquad (\text{A/m}^2). \tag{5–17}$$

Equation (5–17) states that the convection current density in a space-charge limited vacuum diode is proportional to the three-halves power of the potential difference between the anode and the cathode. This nonlinear relation is known as the ***Child-Langmuir law***. ▬

In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes, and ions) drifting with different velocities. Equation (5–3) should be generalized to read

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\text{A/m}^2). \tag{5–18}$$

As indicated in Section 5–1, conduction currents are the result of the drift motion of charge carriers under the influence of an applied electric field. The atoms remain neutral ($\rho = 0$). It can be justified analytically that for most conducting materials the average drift velocity is directly proportional to the electric field intensity. For metallic conductors we write

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\text{m/s}), \tag{5–19}$$

where μ_e is the electron ***mobility*** measured in (m²/V·s). The electron mobility for copper is 3.2×10^{-3} (m²/V·s). It is 1.4×10^{-4} (m²/V·s) for aluminum and 5.2×10^{-3}

(m^2/V·s) for silver. From Eqs. (5–3) and (5–19) we have

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}, \tag{5–20}$$

where $\rho_e = -Ne$ is the charge density of the drifting electrons and is a negative quantity. Equation (5–20) can be rewritten as

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2),} \tag{5–21}$$

where the proportionality constant, $\sigma = -\rho_e \mu_e$, is a macroscopic constitutive parameter of the medium called ***conductivity***.

For semiconductors, conductivity depends on the concentration and mobility of both electrons and holes:

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h, \tag{5–22}$$

where the subscript h denotes hole. In general, $\mu_e \neq \mu_h$. For germanium, typical values are $\mu_e = 0.38$, $\mu_h = 0.18$; for silicon, $\mu_e = 0.12$, $\mu_h = 0.03$ (m^2/V·s).

Equation (5–21) is a constitutive relation of a conducting medium. Isotropic materials for which the linear relation Eq. (5–21) holds are called ***ohmic media***. The unit for σ is ampere per volt-meter (A/V·m) or siemens per meter (S/m). Copper, the most commonly used conductor, has a conductivity 5.80×10^7 (S/m). On the other hand, the conductivity of germanium is around 2.2 (S/m), and that of silicon is 1.6×10^{-3} (S/m). The conductivity of semiconductors is highly dependent of (increases with) temperature. Hard rubber, a good insulator, has a conductivity of only 10^{-15} (S/m). Appendix B–4 lists the conductivities of some other frequently used materials. However, note that, unlike the dielectric constant, the conductivity of materials varies over an extremely wide range. The reciprocal of conductivity is called ***resistivity***, in ohm-meters (Ω·m). We prefer to use conductivity; there is really no compelling need to use both conductivity and resistivity.

We recall ***Ohm's law*** from circuit theory that the voltage V_{12} across a resistance R, in which a current I flows from point 1 to point 2, is equal to RI; that is,

$$V_{12} = RI. \tag{5–23}$$

Here R is usually a piece of conducting material of a given length; V_{12} is the voltage between two terminals 1 and 2; and I is the total current flowing from terminal 1 to terminal 2 through a finite cross section.

Equation (5–23) is *not* a point relation. Although there is little resemblance between Eq. (5–21) and Eq. (5–23), the former is generally referred to as the ***point form of Ohm's law***. It holds at all points in space, and σ can be a function of space coordinates.

Let us use the point form of Ohm's law to derive the voltage-current relationship of a piece of homogeneous material of conductivity σ, length ℓ, and uniform cross section S, as shown in Fig. 5–3. Within the conducting material, $\mathbf{J} = \sigma\mathbf{E}$, where both $\mathbf{J}$ and $\mathbf{E}$ are in the direction of current flow. The potential difference or voltage

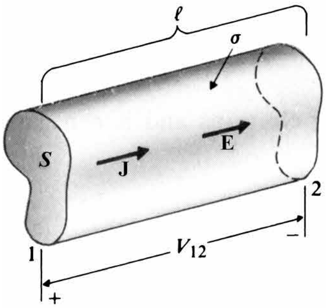


FIGURE 5-3
Homogeneous conductor with a constant cross section.

between terminals 1 and 2 is†

$$V_{12} = E\ell$$

or

$$E = \frac{V_{12}}{\ell}. \tag{5-24}$$

The total current is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS$$

or

$$J = \frac{I}{S}. \tag{5-25}$$

Using Eqs. (5–24) and (5–25) in Eq. (5–21), we obtain

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

or

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI, \tag{5-26}$$

which is the same as Eq. (5–23). From Eq. (5–26) we have the formula for the ***resistance*** of a straight piece of homogeneous material of a uniform cross section for steady current (d.c.):

$$\boxed{R = \frac{\ell}{\sigma S} \qquad (\Omega).} \tag{5-27}$$

We could have started with Eq. (5–23) as the experimental Ohm's law and applied it to a homogeneous conductor of length ℓ and uniform cross-section S. Using the formula in Eq. (5–27), we could derive the point relationship in Eq. (5–21).

† We will discuss the significance of V_{12} and E more in detail in Section 5–3.

EXAMPLE 5–2 Determine the d-c resistance of 1-(km) of wire having a 1-(mm) radius (a) if the wire is made of copper, and (b) if the wire is made of aluminum.

Solution Since we are dealing with conductors of a uniform cross section, Eq. (5–27) applies.

a) For copper wire, $\sigma_{cu} = 5.80 \times 10^7$ (S/m):

$$\ell = 10^3 \text{ (m)}, \qquad S = \pi(10^{-3})^2 = 10^{-6}\pi \quad (\text{m}^2).$$

We have

$$R_{cu} = \frac{\ell}{\sigma_{cu} S} = \frac{10^3}{5.80 \times 10^7 \times 10^{-6}\pi} = 5.49 \quad (\Omega).$$

b) For aluminum wire, $\sigma_{al} = 3.54 \times 10^7$ (S/m):

$$R_{al} = \frac{\ell}{\sigma_{al} S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49 = 8.99 \quad (\Omega).$$

The ***conductance***, G, or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \quad (\text{S}). \tag{5–28}$$

From circuit theory we know the following:

a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

$$\boxed{R_{sr} = R_1 + R_2.} \tag{5–29}$$

b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\boxed{\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}} \tag{5–30a}$$

or

$$\boxed{G_{||} = G_1 + G_2.} \tag{5–30b}$$

5–3 Electromotive Force and Kirchhoff's Voltage Law

In Section 3–2 we pointed out that static electric field is conservative and that the scalar line integral of static electric intensity around any closed path is zero; that is,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0. \tag{5–31}$$

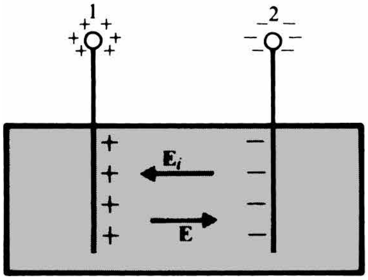


FIGURE 5–4
Electric fields inside an electric battery.

For an ohmic material $\mathbf{J} = \sigma\mathbf{E}$, Eq. (5–31) becomes

$$\oint_C \frac{1}{\sigma}\mathbf{J}\cdot d\boldsymbol{\ell} = 0. \tag{5–32}$$

Equation (5–32) tells us that ***a steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field.*** A steady current in a circuit is the result of the motion of charge carriers, which, in their paths, collide with atoms and dissipate energy in the circuit. This energy must come from a nonconservative field, since a charge carrier completing a closed circuit in a conservative field neither gains nor loses energy. The source of the nonconservative field may be electric batteries (conversion of chemical energy to electric energy), electric generators (conversion of mechanical energy to electric energy), thermocouples (conversion of thermal energy to electric energy), photovoltaic cells (conversion of light energy to electric energy), or other devices. These electrical energy sources, when connected in an electric circuit, provide a driving force for the charge carriers. This force manifests itself as an equivalent ***impressed electric field intensity*** $\mathbf{E}_i$.

Consider an electric battery with electrodes 1 and 2, shown schematically in Fig. 5–4. Chemical action creates a cumulation of positive and negative charges at electrodes 1 and 2, respectively. These charges give rise to an electrostatic field intensity **E** both outside and inside the battery. Inside the battery, **E** must be equal in magnitude and opposite in direction to the nonconservative $\mathbf{E}_i$ produced by chemical action, since no current flows in the open-circuited battery and the net force acting on the charge carriers must vanish. The line integral of the impressed field intensity $\mathbf{E}_i$ from the negative to the positive electrode (from electrode 2 to electrode 1 in Fig. 5–4) inside the battery is customarily called the ***electromotive force***† (emf) of the battery. The SI unit for emf is volt, and an emf is *not* a force in newtons. Denoted by $\mathscr{V}$, the electromotive force is a measure of the strength of the nonconservative source. We have

$$\mathscr{V} = \int_2^1 \mathbf{E}_i\cdot d\boldsymbol{\ell} = -\underset{\substack{\text{Inside}\\ \text{the source}}}{\int_2^1 \mathbf{E}\cdot d\boldsymbol{\ell}}. \tag{5–33}$$

† Also called ***electromotance***.

The conservative electrostatic field intensity $\mathbf{E}$ satisfies Eq. (5–31):

$$\oint_C \mathbf{E} \cdot d\ell = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\substack{\text{Outside} \\ \text{the source}}} + \underbrace{\int_2^1 \mathbf{E} \cdot d\ell}_{\substack{\text{Inside} \\ \text{the source}}} = 0. \tag{5-34}$$

Combining Eqs. (5–33) and (5–34), we have

$$\mathscr{V} = \underbrace{\int_1^2 \mathbf{E} \cdot d\ell}_{\substack{\text{Outside} \\ \text{the source}}} \tag{5-35}$$

or

$$\mathscr{V} = V_{12} = V_1 - V_2. \tag{5-36}$$

In Eqs. (5–35) and (5–36) we have expressed the emf of the source as a line integral of the conservative $\mathbf{E}$ and interpreted it as a ***voltage rise***. In spite of the nonconservative nature of $\mathbf{E}_i$, the emf can be expressed as a potential difference between the positive and negative terminals. This was what we did in arriving at Eq. (5–24).

When a resistor in the form of Fig. 5–3 is connected between terminals 1 and 2 of the battery, completing the circuit, the *total* electric field intensity (electrostatic $\mathbf{E}$ caused by charge cumulation, as well as impressed $\mathbf{E}_i$ caused by chemical action), must be used in the point form of Ohm's law. We have, instead of Eq. (5–21),

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i), \tag{5-37}$$

where $\mathbf{E}_i$ exists inside the battery only, while $\mathbf{E}$ has a nonzero value both inside and outside the source. From Eq. (5–37) we obtain

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}. \tag{5-38}$$

The scalar line integral of Eq. (5–38) around the closed circuit yields, in view of Eqs. (5–31) and (5–33),

$$\mathscr{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\ell = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell. \tag{5-39}$$

Equation (5–39) should be compared to Eq. (5–32), which holds when there is no source of nonconservative field. If the resistor has a conductivity σ, length ℓ, and uniform cross section S, $J = I/S$ and the right side of Eq. (5–39) becomes RI. We have[†]

$$\mathscr{V} = RI. \tag{5-40}$$

If there are more than one source of electromotive force and more than one resistor (including the internal resistances of the sources) in the closed path, we generalize

[†] We assume the battery to have a negligible internal resistance; otherwise, its effect must be included in Eq. (5–40). An ***ideal voltage source*** is one whose terminal voltage is equal to its emf and is independent of the current flowing through it. This implies that an ideal voltage source has a zero internal resistance.

Eq. (5–40) to

$$\boxed{\sum_j \mathscr{V}_j = \sum_k R_k I_k \qquad \text{(V)}.} \tag{5–41}$$

Equation (5–41) is an expression of ***Kirchhoff's voltage law***. It states that, ***around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances***. It applies to *any closed path* in a network. The direction of tracing the path can be arbitrarily assigned, and the currents in the different resistances need not be the same. Kirchhoff's voltage law is the basis for loop analysis in circuit theory.

5–4 Equation of Continuity and Kirchhoff's Current Law

The ***principle of conservation of charge*** is one of the fundamental postulates of physics. Electric charges may not be created or destroyed; all charges either at rest or in motion must be accounted for at all times. Consider an arbitrary volume V bounded by surface S. A net charge Q exists within this region. If a net current I flows across the surface *out* of this region, the charge in the volume must *decrease* at a rate that equals the current. Conversely, if a net current flows across the surface *into* the region, the charge in the volume must *increase* at a rate equal to the current. The current leaving the region is the total outward flux of the current density vector through the surface S. We have

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dv. \tag{5–42}$$

Divergence theorem, Eq. (2–115), may be invoked to convert the surface integral of $\mathbf{J}$ to the volume integral of $\nabla \cdot \mathbf{J}$. We obtain, for a stationary volume,

$$\int_V \nabla \cdot \mathbf{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} \, dv. \tag{5–43}$$

In moving the time derivative of ρ inside the volume integral, it is necessary to use partial differentiation because ρ may be a function of time as well as of space coordinates. Since Eq. (5–43) must hold regardless of the choice of V, the integrands must be equal. Thus we have

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\text{A/m}^3).} \tag{5–44}$$

This point relationship derived from the principle of conservation of charge is called the ***equation of continuity***.

For steady currents, charge density does not vary with time, $\partial\rho/\partial t = 0$. Equation (5–44) becomes

$$\nabla \cdot \mathbf{J} = 0. \tag{5–45}$$

Thus, steady electric currents are divergenceless or solenoidal. Equation (5–45) is a point relationship and holds also at points where $\rho = 0$ (no flow source). It means that the field lines or streamlines of steady currents close upon themselves, unlike those of electrostatic field intensity that originate and end on charges. Over any enclosed surface, Eq. (5–45) leads to the following integral form:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0, \tag{5–46}$$

which can be written as

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}} \tag{5–47}$$

Equation (5–47) is an expression of ***Kirchhoff's current law***. It states that ***the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero.***[†] Kirchhoff's current law is the basis for node analysis in circuit theory.

In Section 3–6, we stated that charges introduced in the interior of a conductor will move to the conductor surface and redistribute themselves in such a way as to make $\rho = 0$ and $\mathbf{E} = 0$ inside under equilibrium conditions. We are now in a position to prove this statement and to calculate the time it takes to reach an equilibrium. Combining Ohm's law, Eq. (5–21), with the equation of continuity and assuming a constant σ, we have

$$\sigma\nabla \cdot \mathbf{E} = -\frac{\partial\rho}{\partial t}. \tag{5–48}$$

In a simple medium, $\nabla \cdot \mathbf{E} = \rho/\epsilon$, and Eq. (5–48) becomes

$$\frac{\partial\rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0. \tag{5–49}$$

The solution of Eq. (5–49) is

$$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\text{C/m}^3),} \tag{5–50}$$

where ρ_0 is the initial charge density at $t = 0$. Both ρ and ρ_0 can be functions of the space coordinates, and Eq. (5–50) says that the charge density at a given location will decrease with time exponentially. An initial charge density ρ_0 will decay to $1/e$

[†] This includes the currents of current generators at the junction, if any. An ***ideal current generator*** is one whose current is independent of its terminal voltage. This implies that an ideal current source has an infinite internal resistance.

or 36.8% of its value in a time equal to

$$\tau = \frac{\epsilon}{\sigma} \qquad \text{(s)}. \tag{5-51}$$

The time constant τ is called the ***relaxation time***. For a good conductor such as copper—$\sigma = 5.80 \times 10^7$ (S/m), $\epsilon \cong \epsilon_0 = 8.85 \times 10^{-12}$ (F/m)—τ equals 1.52×10^{-19} (s), a very short time indeed. The transient time is so brief that, for all practical purposes, ρ can be considered zero in the interior of a conductor—see Eq. (3–69) in Section 3–6. The relaxation time for a good insulator is not infinite but can be hours or days.

5–5 Power Dissipation and Joule's Law

In Section 5–1 we indicated that under the influence of an electric field, conduction electrons in a conductor undergo a drift motion macroscopically. Microscopically, these electrons collide with atoms on lattice sites. Energy is thus transmitted from the electric field to the atoms in thermal vibration. The work Δw done by an electric field $\mathbf{E}$ in moving a charge q a distance $\Delta\ell$ is $q\mathbf{E} \cdot (\Delta\ell)$, which corresponds to a power

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u}, \tag{5-52}$$

where $\mathbf{u}$ is the drift velocity. The total power delivered to all the charge carriers in a volume dv is

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv,$$

which, by virtue of Eq. (5–18), is

$$dP = \mathbf{E} \cdot \mathbf{J}\, dv$$

or

$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} \qquad (\text{W/m}^3). \tag{5-53}$$

Thus the point function $\mathbf{E} \cdot \mathbf{J}$ is a ***power density*** under steady-current conditions. For a given volume V the total electric power converted into heat is

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \qquad (\text{W}).} \tag{5-54}$$

This is known as ***Joule's law***. (Note that the SI unit for P is watt, not joule, which is the unit for energy or work.) Equation (5–53) is the corresponding point relationship.

In a conductor of a constant cross section, $dv = ds\, d\ell$, with $d\ell$ measured in the direction $\mathbf{J}$. Equation (5–54) can be written as

$$P = \int_L E\, d\ell \int_S J\, ds = VI,$$

where I is the current in the conductor. Since $V = RI$, we have

$$P = I^2R \qquad \text{(W)}. \tag{5–55}$$

Equation (5–55) is, of course, the familiar expression for ohmic power representing the heat dissipated in resistance R per unit time.

5–6 Boundary Conditions for Current Density

When current obliquely crosses an interface between two media with different conductivities, the current density vector changes both in direction and in magnitude. A set of boundary conditions can be derived for **J** in a way similar to that used in Section 3–9 for obtaining the boundary conditions for **D** and **E**. The governing equations for steady current density **J** in the absence of nonconservative energy sources are

Governing Equations for Steady Current Density		
Differential Form	Integral Form	
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$	(5–56)
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma}\mathbf{J} \cdot d\boldsymbol{\ell} = 0$	(5–57)

The divergence equation is the same as Eq. (5–45), and the curl equation is obtained by combining Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$) with $\nabla \times \mathbf{E} = 0$. By applying Eqs. (5–56) and (5–57) at the interface between two ohmic media with conductivities σ_1 and σ_2, we obtain the boundary conditions for the normal and tangential components of **J**.

Without actually constructing a pillbox at the interface as was done in Fig. 3–23, we know from Section 3–9 that ***the normal component of a divergenceless vector field is continuous.*** Hence from $\nabla \cdot \mathbf{J} = 0$ we have

$$J_{1n} = J_{2n} \qquad (\text{A/m}^2). \tag{5–58}$$

Similarly, ***the tangential component of a curl-free vector field is continuous across an interface.*** We conclude from $\nabla \times (\mathbf{J}/\sigma) = 0$ that

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}. \tag{5–59}$$

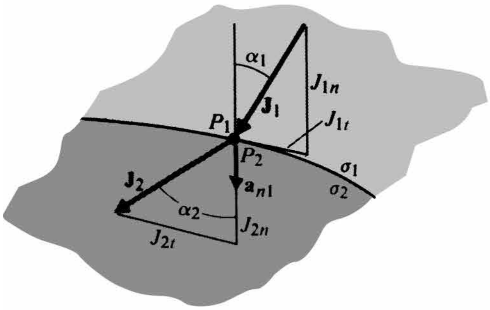


FIGURE 5–5
Boundary conditions at interface between two conducting media (Example 5–3).

Equation (5–59) states that ***the ratio of the tangential components of* J *at two sides of an interface is equal to the ratio of the conductivities.*** Comparing the boundary conditions Eqs. (5–58) and (5–59) for steady current density in ohmic media with the boundary conditions Eqs. (3–123) and (3–119), respectively, for electrostatic flux density at an interface of dielectric media where there are no free charges, we note an exact analogy of **J** and σ with **D** and ϵ.

EXAMPLE 5–3 Two conducting media with conductivities σ_1 and σ_2 are separated by an interface, as shown in Fig. 5–5. The steady current density in medium 1 at point P_1 has a magnitude J_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the current density at point P_2 in medium 2.

Solution Using Eqs. (5–58) and (5–59), we have

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2 \tag{5–60}$$

and

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2. \tag{5–61}$$

Division of Eq. (5–61) by Eq. (5–60) yields

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.} \tag{5–62}$$

If medium 1 is a much better conductor than medium 2 ($\sigma_1 \gg \sigma_2$ or $\sigma_2/\sigma_1 \to 0$), α_2 approaches zero, and $\mathbf{J}_2$ emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of $\mathbf{J}_2$ is

$$\begin{aligned} J_2 &= \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2 \sin \alpha_2)^2 + (J_2 \cos \alpha_2)^2} \\ &= \left[\left(\frac{\sigma_2}{\sigma_1} J_1 \sin \alpha_1\right)^2 + (J_1 \cos \alpha_1)^2\right]^{1/2} \end{aligned}$$

or

$$\boxed{J_2 = J_1\left[\left(\frac{\sigma_2}{\sigma_1}\sin\alpha_1\right)^2 + \cos^2\alpha_1\right]^{1/2}.} \tag{5-63}$$

By examining Fig. 5-5, can you tell whether medium 1 or medium 2 is the better conductor? ▬

For a homogeneous conducting medium the differential form of Eq. (5-57) simplifies to

$$\nabla \times \mathbf{J} = 0. \tag{5-64}$$

From Section 2-11 we know that a curl-free vector field can be expressed as the gradient of a scalar potential field. Let us write

$$\mathbf{J} = -\nabla\psi. \tag{5-65}$$

Substitution of Eq. (5-65) into $\nabla \cdot \mathbf{J} = 0$ yields a Laplace's equation in ψ; that is,

$$\nabla^2\psi = 0. \tag{5-66}$$

A problem in steady-current flow can therefore be solved by determining ψ (A/m) from Eq. (5-66), subject to appropriate boundary conditions and then by finding **J** from its negative gradient in exactly the same way as a problem in electrostatics is solved. As a matter of fact, ψ and electrostatic potential are simply related: $\psi = \sigma V$. As indicated in Section 5-1, this similarity between electrostatic and steady-current fields is the basis for using an electrolytic tank to map the potential distribution of difficult-to-solve electrostatic boundary-value problems.†

When a steady current flows across the boundary between two different lossy dielectrics (dielectrics with permittivities ϵ_1 and ϵ_2 and finite conductivities σ_1 and σ_2), the tangential component of the electric field is continuous across the interface as usual; that is, $E_{2t} = E_{1t}$, which is equivalent to Eq. (5-59). The normal component of the electric field, however, must simultaneously satisfy both Eq. (5-58) and Eq. (3-121b). We require

$$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}, \tag{5-67}$$

$$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s, \tag{5-68}$$

where the reference unit normal is *outward from medium 2*. Hence, unless $\sigma_2/\sigma_1 = \epsilon_2/\epsilon_1$, a surface charge must exist at the interface. From Eqs. (5-67) and (5-68) we find

$$\rho_s = \left(\epsilon_1\frac{\sigma_2}{\sigma_1} - \epsilon_2\right)E_{2n} = \left(\epsilon_1 - \epsilon_2\frac{\sigma_1}{\sigma_2}\right)E_{1n}. \tag{5-69}$$

† See, for instance, E. Weber, *Electromagnetic Fields*, John Wiley and Sons, 1950, Vol. I: *Mapping of Fields*, pp. 187-193.

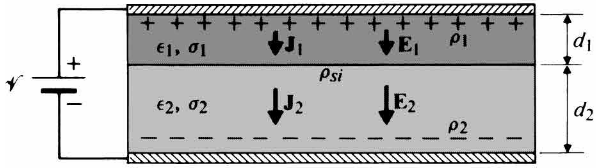


FIGURE 5–6
Parallel-plate capacitor with two lossy dielectrics (Example 5–4).

Again, if medium 2 is a much better conductor than medium 1 ($\sigma_2 \gg \sigma_1$ or $\sigma_1/\sigma_2 \to 0$), Eq. (5–69) becomes approximately

$$\rho_s = \epsilon_1 E_{1n} = D_{1n}, \tag{5–70}$$

which is the same as Eq. (3–122).

EXAMPLE 5–4 An emf $\mathscr{V}$ is applied across a parallel-plate capacitor of area S. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities σ_1 and σ_2, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.

Solution Refer to Fig. 5–6.

a) The continuity of the normal component of **J** assures that the current densities and therefore the currents in both media are the same. By Kirchhoff's voltage law we have

$$\mathscr{V} = (R_1 + R_2)I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right)I.$$

Hence,

$$J = \frac{I}{S} = \frac{\mathscr{V}}{(d_1/\sigma_1) + (d_2/\sigma_2)} = \frac{\sigma_1\sigma_2\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \qquad (\text{A/m}^2). \tag{5–71}$$

b) To determine the electric field intensities E_1 and E_2 in both media, two equations are needed. Neglecting fringing effect at the edges of the plates, we have

$$\mathscr{V} = E_1 d_1 + E_2 d_2 \tag{5–72}$$

and

$$\sigma_1 E_1 = \sigma_2 E_2. \tag{5–73}$$

Equation (5–73) comes from $J_1 = J_2$. Solving Eqs. (5–72) and (5–73), we obtain

$$E_1 = \frac{\sigma_2\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \qquad (\text{V/m}) \tag{5–74}$$

and

$$E_2 = \frac{\sigma_1\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \qquad (\text{V/m}). \tag{5–75}$$

c) The surface charge densities on the upper and lower plates can be determined by using Eq. (5–70):

$$\rho_{s1} = \epsilon_1 E_1 = \frac{\epsilon_1 \sigma_2 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2) \tag{5-76}$$

$$\rho_{s2} = -\epsilon_2 E_2 = -\frac{\epsilon_2 \sigma_1 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2). \tag{5-77}$$

The negative sign in Eq. (5–77) comes about because $\mathbf{E}_2$ and the *outward* normal at the lower plate are in opposite directions.

Equation (5–69) can be used to find the surface charge density at the interface of the dielectrics. We have

$$\begin{aligned} \rho_{si} &= \left(\epsilon_2 \frac{\sigma_1}{\sigma_2} - \epsilon_1\right) \frac{\sigma_2 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \\ &= \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2)\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2). \end{aligned} \tag{5-78}$$

From these results we see that $\rho_{s2} \neq -\rho_{s1}$ but that $\rho_{s1} + \rho_{s2} + \rho_{si} = 0$. ■

In Example 5–4 we encounter a situation in which both static charges and a steady current exist. As we shall see in Chapter 6, a steady current gives rise to a steady magnetic field. We have, then, both a static electric field and a steady magnetic field. They constitute an ***electromagnetostatic field.*** The electric and magnetic fields of an electromagnetostatic field are coupled through the constitutive relation $\mathbf{J} = \sigma \mathbf{E}$ of the conducting medium.

5–7 Resistance Calculations

In Section 3–10 we discussed the procedure for finding the capacitance between two conductors separated by a dielectric medium. These conductors may be of arbitrary shapes, as was shown in Fig. 3–27, which is reproduced here as Fig. 5–7. In terms of electric field quantities the basic formula for capacitance can be written as

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}, \tag{5-79}$$

where the surface integral in the numerator is carried out over a surface enclosing the positive conductor and the line integral in the denominator is from the negative (lower-potential) conductor to the positive (higher-potential) conductor (see Eq. 5–35).

When the dielectric medium is lossy (having a small but nonzero conductivity), a current will flow from the positive to the negative conductor, and a current-density field will be established in the medium. Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, ensures that the streamlines for $\mathbf{J}$ and $\mathbf{E}$ will be the same in an isotropic medium. The resistance between

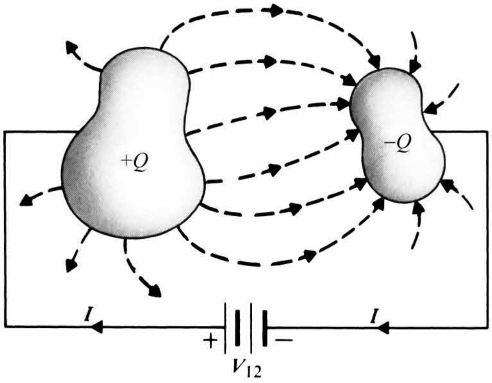


FIGURE 5–7
Two conductors in a lossy dielectric medium.

the conductors is

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}}, \tag{5-80}$$

where the line and surface integrals are taken over the same L and S as those in Eq. (5–79). Comparison of Eqs. (5–79) and (5–80) shows the following interesting relationship:

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.} \tag{5-81}$$

Equation (5–81) holds if ϵ and σ of the medium have the same space dependence or if the medium is homogeneous (independent of space coordinates). In these cases, if the capacitance between two conductors is known, the resistance (or conductance) can be obtained directly from the ϵ/σ ratio without recomputation.

EXAMPLE 5–5 Find the leakage resistance per unit length (a) between the inner and outer conductors of a coaxial cable that has an inner conductor of radius a, an outer conductor of inner radius b, and a medium with conductivity σ, and (b) of a parallel-wire transmission line consisting of wires of radius a separated by a distance D in a medium with conductivity σ.

Solution

a) The capacitance per unit length of a coaxial cable has been obtained as Eq. (3–139) in Example 3–18:

$$C_1 = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{(F/m)}.$$

Hence the leakage resistance per unit length is, from Eq. (5–81),

$$R_1 = \frac{\epsilon}{\sigma}\left(\frac{1}{C_1}\right) = \frac{1}{2\pi\sigma}\ln\left(\frac{b}{a}\right) \qquad (\Omega\cdot\text{m}). \tag{5–82}$$

The conductance per unit length is $G_1 = 1/R_1$.

b) For the parallel-wire transmission line, Eq. (4–47) in Example 4–4 gives the capacitance per unit length:

$$C_1' = \frac{\pi\epsilon}{\cosh^{-1}\left(\dfrac{D}{2a}\right)} \qquad (\text{F/m}).$$

Therefore the leakage resistance per unit length is, without further ado,

$$\begin{aligned} R_1' &= \frac{\epsilon}{\sigma}\left(\frac{1}{C_1'}\right) = \frac{1}{\pi\sigma}\cosh^{-1}\left(\frac{D}{2a}\right) \\ &= \frac{1}{\pi\sigma}\ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right] \qquad (\Omega\cdot\text{m}). \end{aligned} \tag{5–83}$$

The conductance per unit length is $G_1' = 1/R_1'$. ∎

It must be emphasized here that the resistance *between* the conductors for a length ℓ of the coaxial cable is R_1/ℓ, not ℓR_1; similarly, the leakage resistance of a length ℓ of the parallel-wire transmission line is R_1'/ℓ, not $\ell R_1'$. Do you know why?

In certain situations, electrostatic and steady-current problems are not exactly analogous, even when the geometrical configurations are the same. This is because current flow can be confined strictly within a conductor (which has a *very large* σ in comparison to that of the surrounding medium), whereas electric flux usually cannot be contained within a dielectric slab of finite dimensions. The range of the dielectric constant of available materials is very limited (see Appendix B–3), and the flux-fringing around conductor edges makes the computation of capacitance less accurate.

The procedure for computing the resistance of a piece of conducting material between specified equipotential surfaces (or terminals) is as follows:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a potential difference V_0 between conductor terminals.
3. Find electric field intensity $\mathbf{E}$ within the conductor. (If the material is homogeneous, having a *constant* conductivity, the general method is to solve Laplace's equation $\nabla^2 V = 0$ for V in the chosen coordinate system, and then obtain $\mathbf{E} = -\nabla V$.)
4. Find the total current
$$I = \int_S \mathbf{J}\cdot d\mathbf{s} = \int_S \sigma\mathbf{E}\cdot d\mathbf{s},$$
where S is the cross-sectional area over which I flows.
5. Find resistance R by taking the ratio V_0/I.

It is important to note that if the conducting material is inhomogeneous and if the conductivity is a function of space coordinates, Laplace's equation for V does not hold. Can you explain why and indicate how $\mathbf{E}$ can be determined under these circumstances?

When the given geometry is such that $\mathbf{J}$ can be determined easily from a total current I, we may start the solution by assuming an I. From I, $\mathbf{J}$ and $\mathbf{E} = \mathbf{J}/\sigma$ are found. Then the potential difference V_0 is determined from the relation

$$V_0 = -\int \mathbf{E} \cdot d\boldsymbol{\ell},$$

where the integration is from the low-potential terminal to the high-potential terminal. The resistance $R = V_0/I$ is independent of the assumed I, which will be canceled in the process.

EXAMPLE 5–6 A conducting material of uniform thickness h and conductivity σ has the shape of a quarter of a flat circular washer, with inner radius a and outer radius b, as shown in Fig. 5–8. Determine the resistance between the end faces.

Solution Obviously, the appropriate coordinate system to use for this problem is the cylindrical coordinate system. Following the foregoing procedure, we first assume a potential difference V_0 between the end faces, say $V = 0$ on the end face at $y = 0$ ($\phi = 0$) and $V = V_0$ on the end face at $x = 0$ ($\phi = \pi/2$). We are to solve Laplace's equation in V subject to the following boundary conditions:

$$V = 0 \quad \text{at} \quad \phi = 0, \tag{5–84a}$$

$$V = V_0 \quad \text{at} \quad \phi = \pi/2. \tag{5–84b}$$

Since potential V is a function of ϕ only, Laplace's equation in cylindrical coordinates simplifies to

$$\frac{d^2V}{d\phi^2} = 0. \tag{5–85}$$

The general solution of Eq. (5–85) is

$$V = c_1\phi + c_2,$$

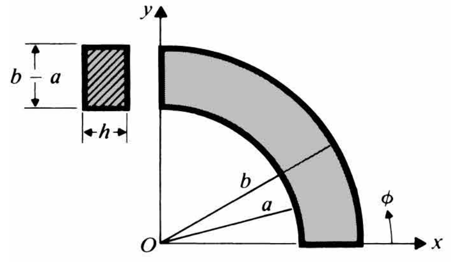


FIGURE 5–8
A quarter of a flat conducting circular washer (Example 5–6).

which, upon using the boundary conditions in Eqs. (5–84a) and (5–84b), becomes

$$V = \frac{2V_0}{\pi}\phi. \tag{5–86}$$

The current density is

$$\begin{aligned}\mathbf{J} &= \sigma\mathbf{E} = -\sigma\nabla V \\ &= -\mathbf{a}_\phi\sigma\frac{\partial V}{r\partial\phi} = -\mathbf{a}_\phi\frac{2\sigma V_0}{\pi r}.\end{aligned} \tag{5–87}$$

The total current I can be found by integrating $\mathbf{J}$ over the $\phi = \pi/2$ surface at which $d\mathbf{s} = -\mathbf{a}_\phi h\,dr$. We have

$$\begin{aligned}I &= \int_S \mathbf{J}\cdot d\mathbf{s} = \frac{2\sigma V_0}{\pi}h\int_a^b \frac{dr}{r} \\ &= \frac{2\sigma h V_0}{\pi}\ln\frac{b}{a}.\end{aligned} \tag{5–88}$$

Therefore,

$$R = \frac{V_0}{I} = \frac{\pi}{2\sigma h\ln(b/a)}. \tag{5–89}$$

Note that, for this problem, it is not convenient to begin by assuming a total current I because it is not obvious how $\mathbf{J}$ varies with r for a given I. Without $\mathbf{J}$, $\mathbf{E}$ and V_0 cannot be determined. ▬

Review Questions

R.5–1 Explain the difference between conduction and convection currents.

R.5–2 Explain the operation of an electrolytic tank. In what ways do electrolytic currents differ from conduction and convection currents?

R.5–3 Define *mobility* of the electron in a conductor. What is its SI unit?

R.5–4 What is the *Child-Langmuir law*?

R.5–5 What is the point form for *Ohm's law*?

R.5–6 Define *conductivity*. What is its SI unit?

R.5–7 Why does the resistance formula in Eq. (5–27) require that the material be homogeneous and straight and that it have a uniform cross section?

R.5–8 Prove Eqs. (5–29) and (5–30b).

R.5–9 Define *electromotive force* in words.

R.5–10 What is the difference between impressed and electrostatic field intensities?

R.5–11 State *Kirchhoff's voltage law* in words.

R.5–12 What are the characteristics of an ideal voltage source?

R.5–13 Can the currents in different branches (resistors) of a closed loop in an electric network flow in opposite directions? Explain.

R.5–14 What is the physical significance of the *equation of continuity*?

R.5–15 State *Kirchhoff's current law* in words.

R.5–16 What are the characteristics of an ideal current source?

R.5–17 Define *relaxation time.* What is the order of magnitude of the relaxation time in copper?

R.5–18 In what ways should Eq. (5–48) be modified when σ is a function of space coordinates?

R.5–19 State Joule's law. Express the power dissipated in a volume

a) in terms of **E** and σ,

b) in terms of **J** and σ.

R.5–20 Does the relation $\nabla \times \mathbf{J} = 0$ hold in a medium whose conductivity is not constant? Explain.

R.5–21 What are the boundary conditions of the normal and tangential components of steady current at the interface of two media with different conductivities?

R.5–22 What quantities in electrostatics are analogous to the steady current density vector and conductivity in an ohmic medium?

R.5–23 What is the basis of using an electrolytic tank to map the potential distribution of electrostatic boundary-value problems?

R.5–24 What is the relation between the resistance and the capacitance formed by two conductors immersed in a lossy dielectric medium that has permittivity ϵ and conductivity σ?

R.5–25 Under what situations will the relation between R and C in R.5–24 be only approximately correct? Give a specific example.

Problems

P.5–1 Assuming S to be the area of the electrodes in the space-charge-limited vacuum diode in Fig. 5–2, find

a) $V(y)$ and $E(y)$ within the interelectrode region,

b) the total amount of charge in the interelectrode region,

c) the total surface charge on the cathode and on the anode,

d) the transit time of an electron from the cathode to the anode with $V_0 = 200$ (V) and $d = 1$ (cm).

P.5–2 Starting with Ohm's law as expressed in Eq. (5–26) applied to a resistor of length ℓ, conductivity σ, and uniform cross-section S, verify the point form of Ohm's law represented by Eq. (5–21).

P.5–3 A long, round wire of radius a and conductivity σ is coated with a material of conductivity 0.1σ.

a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50%?

b) Assuming a total current I in the coated wire, find **J** and **E** in both the core and the coating material.

P.5–4 Find the current and the heat dissipated in each of the five resistors in the network shown in Fig. 5–9 if

$$R_1 = \tfrac{1}{3}\,(\Omega), \quad R_2 = 20\,(\Omega), \quad R_3 = 30\,(\Omega), \quad R_4 = 8\,(\Omega), \quad R_5 = 10\,(\Omega),$$

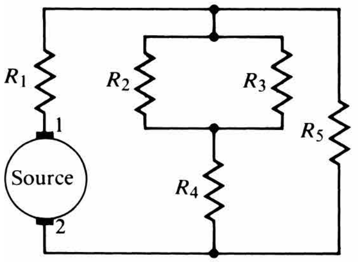


FIGURE 5–9
A network problem (Problem P.5–4).

and if the source is an ideal d-c voltage generator of 0.7 (V) with its positive polarity at terminal 1. What is the total resistance seen by the source at terminal pair 1–2?

P.5–5 Solve Problem P.5–4, assuming that the source is an ideal current generator that supplies a direct current of 0.7 (A) out of terminal 1.

P.5–6 Lightning strikes a lossy dielectric sphere—$\epsilon = 1.2\,\epsilon_0$, $\sigma = 10$ (S/m)—of radius 0.1 (m) at time $t = 0$, depositing uniformly in the sphere a total charge 1 (mC). Determine, for all t,

a) the electric field intensity both inside and outside the sphere,
b) the current density in the sphere.

P.5–7 Refer to Problem P.5–6.

a) Calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value.
b) Calculate the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its value. What happens to this energy?
c) Determine the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

P.5–8 A d-c voltage of 6 (V) applied to the ends of 1 (km) of a conducting wire of 0.5 (mm) radius results in a current of 1/6 (A). Find

a) the conductivity of the wire,
b) the electric field intensity in the wire,
c) the power dissipated in the wire,
d) the electron drift velocity, assuming electron mobility in the wire to be 1.4×10^{-3} $(\text{m}^2/\text{V}\cdot\text{s})$.

P.5–9 Two lossy dielectric media with permittivities and conductivities (ϵ_1, σ_1) and (ϵ_2, σ_2) are in contact. An electric field with a magnitude E_1 is incident from medium 1 upon the interface at an angle α_1 measured from the common normal, as in Fig. 5–10.

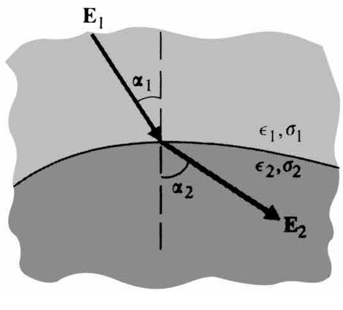


FIGURE 5–10
Boundary between two lossy dielectric media (Problem P.5–9).

a) Find the magnitude and direction of $\mathbf{E}_2$ in medium 2.
b) Find the surface charge density at the interface.
c) Compare the results in parts (a) and (b) with the case in which both media are perfect dielectrics.

P.5–10 The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as in Fig. 5–11. Determine
a) the total resistance between the plates,
b) the surface charge densities on the plates,
c) the volume charge density and the total amount of charge between the plates.

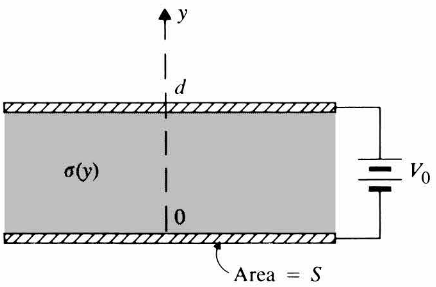


FIGURE 5–11
Inhomogeneous ohmic medium with conductivity $\sigma(y)$ (Problem P.5–10).

P.5–11 Refer to Example 5–4.
a) Draw the equivalent circuit of the two-layer, parallel-plate capacitor with lossy dielectrics, and identify the magnitude of each component.
b) Determine the power dissipated in the capacitor.

P.5–12 Refer again to Example 5–4. Assuming that a voltage V_0 is applied across the parallel-plate capacitor with the two layers of different lossy dielectrics at $t = 0$,
a) express the surface charge density ρ_{si} at the dielectric interface as a function of t,
b) express the electric field intensities $\mathbf{E}_1$ and $\mathbf{E}_2$ as functions of t.

P.5–13 A d-c voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ϵ_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ϵ_2 and conductivity σ_2 in the region $c < r < b$. Determine
a) the current density in each region,
b) the surface charge densities on the inner and outer conductors and at the interface between the two dielectrics.

P.5–14 Refer to the flat conducting quarter-circular washer in Example 5–6 and Fig. 5–8. Find the resistance between the curved sides.

P.5–15 Find the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$) if the space between the surfaces is filled with a homogeneous and isotropic material having a conductivity σ.

P.5–16 Determine the resistance between two concentric spherical surfaces of radii R_1 and R_2 $(R_1 < R_2)$, assuming that a material of conductivity $\sigma = \sigma_0(1 + k/R)$ fills the space between them. (*Note:* Laplace's equation for V does not apply here.)

P.5–17 A homogeneous material of uniform conductivity σ is shaped like a truncated conical block and defined in spherical coordinates by

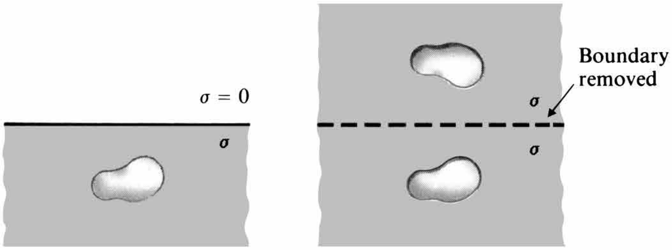
$$R_1 \leq R \leq R_2 \quad \text{and} \quad 0 \leq \theta \leq \theta_0.$$

Determine the resistance between the $R = R_1$ and $R = R_2$ surfaces.

P.5–18 Redo Problem P.5–17, assuming that the truncated conical block is composed of an inhomogeneous material with a nonuniform conductivity $\sigma(R) = \sigma_0 R_1/R$, where $R_1 \leq R \leq R_2$.

P.5–19 Two conducting spheres of radii b_1 and b_2 that have a very high conductivity are immersed in a poorly conducting medium (for example, they are buried very deep in the ground) of conductivity σ and permittivity ϵ. The distance, d, between the spheres is very large in comparison with the radii. Determine the resistance between the conducting spheres. (*Hint:* Find the capacitance between the spheres by following the procedure in Section 3–10 and using Eq. (5–81).)

P.5–20 Justify the statement that the steady-current problem associated with a conductor buried in a poorly conducting medium near a plane boundary with air, as shown in Fig. 5–12(a), can be replaced by that of the conductor and its image, both immersed in the poorly conducting medium as shown in Fig. 5–12(b).



(a) Conductor in a poorly conducting medium near a plane boundary.

(b) Image conductor in conducting medium replacing the plane boundary.

FIGURE 5–12
Steady-current problem with a plane boundary (Problem P.5–20).

P.5–21 A ground connection is made by burying a hemispherical conductor of radius 25 (mm) in the earth with its base up, as shown in Fig. 5–13. Assuming the earth conductivity to be 10^{-6} S/m, find the resistance of the conductor to far-away points in the ground. (*Hint:* Use the image method in P.5–20.)

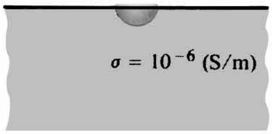


FIGURE 5–13
Hemispherical conductor in ground (Problem P.5–21).

P.5–22 Assume a rectangular conducting sheet of conductivity σ, width a, and height b. A potential difference V_0 is applied to the side edges, as shown in Fig. 5–14. Find

a) the potential distribution,

b) the current density everywhere within the sheet. (*Hint:* Solve Laplace's equation in Cartesian coordinates subject to appropriate boundary conditions.)

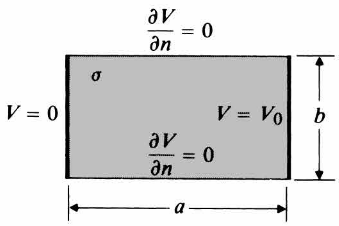


FIGURE 5–14
A conducting sheet (Problem P.5–22).

P.5–23 A uniform current density $\mathbf{J} = \mathbf{a}_x J_0$ flows in a very large rectangular block of homogeneous material of a uniform thickness having a conductivity σ. A hole of radius b is drilled in the material. Find the new current density $\mathbf{J}'$ in the conducting material. (*Hint:* Solve Laplace's equation in cylindrical coordinates and note that V approaches $-(J_0 r/\sigma) \cos \phi$ as $r \to \infty$, where ϕ is the angle measured from the x-axis.)