第5章 基于LMI的控制系统抗饱和控制

在实际的控制系统中，由于其自身的物理特性而引起的执行机构输出幅值是有限的，即输入饱和问题，该问题是目前控制系统中最为常见的一种非线性问题。由于控制输入饱和的存在，可能导致整个控制系统发散，进而导致整个控制系统失控。即使系统不发散，长时间高强度的振荡也会导致控制系统的结构损坏，从而导致故障。控制输入饱和问题是多年来研究的热门课题。

目前解决控制输入受限下控制器的设计问题有两种方法：一种是设计有界的控制输入，例如文献[3,4]中，采用双曲正切函数设计有界的控制律，但该方法过于保守，会造成执行器利用的效率不高；另一种是基于控制输入饱和的方法，采用 LMI 方法设计控制器的增益[1,2]，实现控制输入超出界限部分的有效补偿，该方法与有界控制输入方法相比，可保证控制输入按最大的值，有效地提高了执行器的利用效率。

近些年来国内外的学者对抗饱和问题进行了深入的研究，线性矩阵不等式 LMI 作为一种有效的数学工具，被广泛地应用于抗饱和控制领域之中。G. Grimm 等[2]针对一般情况下的稳定模型系统设计了基于 LMI 的动态补偿器，保证系统稳定，并保证系统输出对外部干扰具有 L_2 增益。H. S. Hu 等[5]针对一般系统研究了基于 LMI 的 L_2 增益特性及稳定区域。

本章针对控制系统抗饱和问题，介绍了基于 LMI 的抗饱和补偿器，保证了系统在输入受限情况下闭环稳定。

5.1 LQG 控制器的设计

5.1.1 系统描述

考虑如下模型

$$\begin{aligned}\dot{\boldsymbol{x}}_{\mathrm{p}} &= \boldsymbol{A}_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{B}_{\mathrm{pu}}u + \boldsymbol{B}_{\mathrm{pw}}\boldsymbol{w} \\ y &= \boldsymbol{C}_{\mathrm{py}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pyu}}u + \boldsymbol{D}_{\mathrm{pyw}}\boldsymbol{w} + v_0\end{aligned} \tag{5.1}$$

其中，$\boldsymbol{x}_{\mathrm{p}} = [x_1 \quad x_2]^{\mathrm{T}}$，$\boldsymbol{w} = [w_0 \quad r]^{\mathrm{T}}$，$\boldsymbol{B}_{\mathrm{pw}} = [\boldsymbol{B}_{\mathrm{pw0}} \quad \boldsymbol{B}_{\mathrm{pwr}}]$，$\boldsymbol{D}_{\mathrm{pyw}} = [\boldsymbol{D}_{\mathrm{pyw0}} \quad \boldsymbol{D}_{\mathrm{pywr}}]$，$\boldsymbol{B}_{\mathrm{pwr}} = 0$，$\boldsymbol{D}_{\mathrm{pywr}} = 0$，$r$ 为指令，w_0 和 v_0 分别为加在输入和输出的信号噪声。

5.1.2 控制器的设计

LQG 控制器由 LQI 控制器和 Kalman 滤波器组合构成，如图 5.1 所示。采用 Kalman 滤波器实现文献[1]中控制器式(4.9)的动态系统结构，同时可实现信号的滤波。

采用带有积分的 LQG 控制器结构如图 5.2 所示，其中 $\boldsymbol{x}_i$ 为跟踪误差的积分。采用 lqi()函数求控制器增益 $\boldsymbol{K}$，采用 Kalman 滤波器实现动态系统的状态估计。

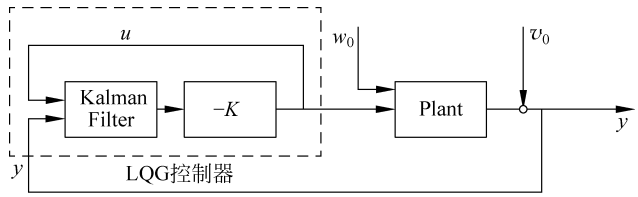


图 5.1 LQG 控制系统结构

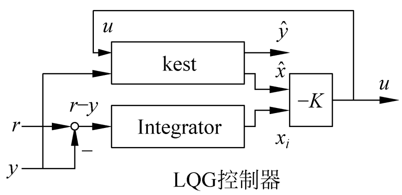


图 5.2 带有积分的 LQG 控制器结构

针对模型式(5.1)，Kalman 滤波算法为

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}_{\mathrm{p}}\hat{\boldsymbol{x}} + \boldsymbol{B}_{\mathrm{pu}}\boldsymbol{u} + \boldsymbol{L}(y - \boldsymbol{C}_{\mathrm{py}}\hat{\boldsymbol{x}} - \boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{u}) \tag{5.2}$$

其中，$\boldsymbol{L}$ 为 kalman 增益。

采用如下命令实现 Kalman 滤波算法，从而求得增益 L：

$$[\mathrm{kest}, \mathrm{L}, \mathrm{P}] = \mathrm{kalman}(\mathrm{sys}, \mathrm{Qn}, \mathrm{Rn}, \mathrm{Nn}) \tag{5.3}$$

其中，sys=ss(Ap,Bpu,Cpy,Dpyu)将模型式(5.1)转换为状态空间的形式，kest 为 Kalman 滤波器的结构信息，Q_{n}，R_{n} 分别表示输入噪声 w_0 和输出噪声 v_0 的协方差数据，$Q_{\mathrm{n}} = \mathrm{E}(w_0 w_0^{\mathrm{T}})$，$R_{\mathrm{n}} = \mathrm{E}(v_0 v_0^{\mathrm{T}})$，$P$ 为 Riccati 方程的信息。

二次型性能指标为

$$J(u) = \int_0^{\infty} \{\bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{Q}\bar{\boldsymbol{x}} + 2\bar{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{u} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u}\}\,\mathrm{d}t \tag{5.4}$$

其中，$\bar{\boldsymbol{x}} = [\hat{\boldsymbol{x}} \quad x_i]^{\mathrm{T}}$，$\boldsymbol{Q}$、$\boldsymbol{R}$、$\boldsymbol{N}$ 分别为描述状态和控制输入的权值矩阵。

采用 MATLAB 中的 lqi()函数，可求得满足二次型性能指标的最优控制增益和控制器为

$$\boldsymbol{K} = \mathrm{lqi}(\mathrm{sys}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{N}) \tag{5.5}$$

$$u = -\boldsymbol{K}\bar{\boldsymbol{x}} = -\boldsymbol{K}_{\mathrm{x}}\hat{\boldsymbol{x}} - K_i x_i \tag{5.6}$$

其中，x_i 为跟踪误差的积分，$\boldsymbol{K} = [\boldsymbol{K}_x \quad K_i]$，$\dot{x}_i = r - y$。

控制目标为：在 $\hat{\boldsymbol{x}}$ 和 $\boldsymbol{u}$ 尽量减小的情况下，保证 $x_i \to 0$，即 $r \to y$。

在 Kalman 滤波器中,由于

$$
\begin{aligned}
&\boldsymbol{A}_{\mathrm{p}}\hat{\boldsymbol{x}}+\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{u}+\boldsymbol{L}(y-C_{\mathrm{py}}\hat{\boldsymbol{x}}-\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{u})\\
&=\boldsymbol{A}_{\mathrm{p}}\hat{\boldsymbol{x}}-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{x}\hat{\boldsymbol{x}}-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{i}\boldsymbol{x}_{i}+\boldsymbol{L}y-\boldsymbol{L}\boldsymbol{C}_{\mathrm{py}}\hat{\boldsymbol{x}}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{x}\hat{\boldsymbol{x}}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{i}\boldsymbol{x}_{i}\\
&=(\boldsymbol{A}_{\mathrm{p}}-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{x}-\boldsymbol{L}\boldsymbol{C}_{\mathrm{py}}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{x})\hat{\boldsymbol{x}}+(-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{i}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{i})\boldsymbol{x}_{i}+\boldsymbol{L}y
\end{aligned}
$$

则 LQG 控制器算法状态方程可写为

$$
\begin{bmatrix}\dot{\hat{\boldsymbol{x}}}\\ \dot{x}_{i}\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_{\mathrm{p}}-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{x}-\boldsymbol{L}\boldsymbol{C}_{\mathrm{py}}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{x} & -\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{i}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{i}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{x}}\\ x_{i}\end{bmatrix}+\begin{bmatrix}0 & \boldsymbol{L}\\ 1 & -1\end{bmatrix}\begin{bmatrix}r\\ y\end{bmatrix} \tag{5.7}
$$

其中,$\begin{bmatrix}0 & \boldsymbol{L}\\ 1 & -1\end{bmatrix}\begin{bmatrix}r\\ y\end{bmatrix}=\begin{bmatrix}\boldsymbol{L}\\ -1\end{bmatrix}y+\begin{bmatrix}0\\ 1\end{bmatrix}r$。

5.1.3 闭环系统的状态方程表示

根据式(5.1),可知被控对象状态方程

$$
\begin{aligned}
\dot{\boldsymbol{x}}_{\mathrm{p}}&=\boldsymbol{A}_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}}+\boldsymbol{B}_{\mathrm{pu}}u+\boldsymbol{B}_{\mathrm{pw0}}w_{0}\\
y&=\boldsymbol{C}_{\mathrm{py}}\boldsymbol{x}_{\mathrm{p}}+\boldsymbol{D}_{\mathrm{pyu}}u+\boldsymbol{D}_{\mathrm{pyw0}}w_{0}+v_{0}
\end{aligned} \tag{5.8}
$$

根据式(5.6)和式(5.7),可将 LQG 控制器算法写成状态方程形式

$$
\begin{aligned}
\dot{\boldsymbol{x}}_{\mathrm{c}}&=\boldsymbol{A}_{\mathrm{c}}\boldsymbol{x}_{\mathrm{c}}+\boldsymbol{B}_{\mathrm{cy}}y+\boldsymbol{B}_{\mathrm{cw}}\boldsymbol{w}\\
u&=\boldsymbol{C}_{\mathrm{c}}\boldsymbol{x}_{\mathrm{c}}+D_{\mathrm{cy}}y+\boldsymbol{D}_{\mathrm{cw}}\boldsymbol{w}
\end{aligned} \tag{5.9}
$$

其中,$\boldsymbol{x}_{\mathrm{c}}=\begin{bmatrix}\hat{\boldsymbol{x}}\\ x_{i}\end{bmatrix}$,$\boldsymbol{A}_{\mathrm{c}}=\begin{bmatrix}\boldsymbol{A}_{\mathrm{p}}-\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{x}-\boldsymbol{L}\boldsymbol{C}_{\mathrm{py}}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{x} & -\boldsymbol{B}_{\mathrm{pu}}\boldsymbol{K}_{i}+\boldsymbol{L}\boldsymbol{D}_{\mathrm{pyu}}\boldsymbol{K}_{i}\\ 0 & 0\end{bmatrix}$,$\boldsymbol{B}_{\mathrm{cy}}=\begin{bmatrix}\boldsymbol{L}\\ -1\end{bmatrix}$,$\boldsymbol{B}_{\mathrm{cw}}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$,$\boldsymbol{C}_{\mathrm{c}}=-\boldsymbol{K}$,$D_{\mathrm{cy}}=0$,$\boldsymbol{D}_{\mathrm{cw}}=[0\quad 0]$。

5.1.4 仿真实例

被控对象取式(5.1),即

$$
\begin{aligned}
\dot{x}_{1}&=x_{2}\\
\dot{x}_{2}&=-x_{1}-10x_{2}+u+w_{0}\\
y&=x_{1}+w_{0}+v_{0}
\end{aligned}
$$

对应式(5.1),$\boldsymbol{A}_{\mathrm{p}}=\begin{bmatrix}0 & 1\\ -1 & -10\end{bmatrix}$,$\boldsymbol{B}_{\mathrm{pu}}=\begin{bmatrix}0\\ 1\end{bmatrix}$,$\boldsymbol{B}_{\mathrm{pw}}=\boldsymbol{B}_{\mathrm{pu}}$,$\boldsymbol{C}_{\mathrm{py}}=[1\quad 0]$,$\boldsymbol{D}_{\mathrm{pyu}}=0$,$\boldsymbol{D}_{\mathrm{pyw0}}=1$,$w_{0}$ 和 v_{0} 分别为加在输入和输出的信号噪声,$w_{0}=0.001\sin(10t)$,$v_{0}=0.001\mathrm{rands}(1)$。

通过 LQI 方法求控制增益 $\boldsymbol{K}$,根据二次型性能指标式(5.4),不考虑 w_{0} 和输出噪声

v_0 之间的相互影响，取 $\boldsymbol{Q}=\begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 60\end{bmatrix}$，$R=0.001$，$N=0$，利用 $\boldsymbol{K}=\text{lqi(sys,Q,R,N)}$ 求得 $\boldsymbol{K}=-\boldsymbol{C}_c=[202.3428\quad 64.1936\quad -244.949]$。

通过 Kalman 滤波算法求增益 $\boldsymbol{L}$，根据式(5.3)，将输入噪声 w_0 和输出噪声 v_0 的协方差数据分别取为 $Q_n=100$，$R_n=100$，利用[kalmf,L]=kalman(Plant,Q_n,R_n)求得增益 $\boldsymbol{L}=\begin{bmatrix}0.008\\0.50\end{bmatrix}$，从而得到控制器式(5.9)。取指令 r 为方波信号，$r=1.5\text{square}(0.25t)$，仿真结果如图 5.3 和图 5.4 所示。

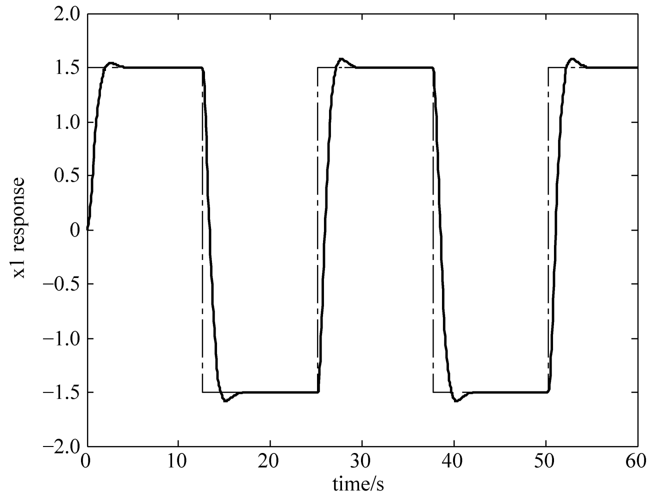


图 5.3　x_1 的方波响应

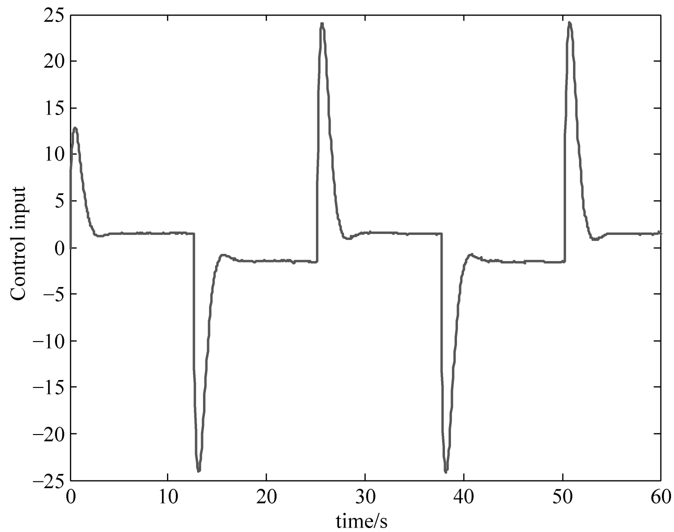


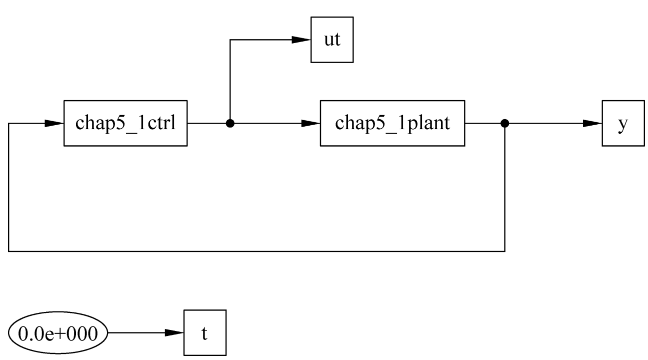
图 5.4　控制输入

仿真程序：

（1）LQG控制器设计程序：chap5_1LQG.m

```
%LQG Controller design
clear all;
close all;

%Plant
Ap=[0 1;-1 -10];
Bpu=[0;1];
Bpw0=Bpu;
Cpy=[1 0];
Dpyu=0;
Dpyw0=1;

%LQI design
sys=ss(Ap,Bpu,Cpy,Dpyu); %in state space without noise
Q=[5 0 0;0 5 0;0 0 60];                        %Used in lqi for x and xi
R=0.001;
N=0;
K=lqi(sys,Q,R,N);

Kx=[K(1) K(2)];
Ki=K(3);
%Kalman filter
Plant=ss(Ap,[Bpu Bpw0],Cpy,[Dpyu Dpyw0]);    %in state space with noise
Qn=100;Rn=100;
[kalmf,L]=kalman(Plant,Qn,Rn);

%LQG controller
Ac=[Ap-Bpu*Kx-L*Cpy+L*Dpyu*Kx  -Bpu*Ki+L*Dpyu*Ki;
    0 0 0]
Bcy=[L;-1];
Bcw=[0 0;0 0;0 1];

Cc=-K
Dcy=0;
Dcw=[0 0];

save lqg_file Ac Bcy Bcw Cc Dcy Dcw;
```

（2）Simulink主程序：chap5_1sim.mdl



(3) 控制器程序：chap5_1ctrl.m

```
function [sys,x0,str,ts] = s_function(t,x,u,flag)
switch flag,
case 0,
    [sys,x0,str,ts] = mdlInitializeSizes;
case 1,
    sys = mdlDerivatives(t,x,u);
case 3,
    sys = mdlOutputs(t,x,u);
case {1,2, 4, 9 }
    sys = [];
otherwise
    error(['Unhandled flag = ',num2str(flag)]);
end
function [sys,x0,str,ts] = mdlInitializeSizes
sizes = simsizes;
sizes.NumContStates  = 3;
sizes.NumDiscStates  = 0;
sizes.NumOutputs     = 1;
sizes.NumInputs      = 1;
sizes.DirFeedthrough = 1;
sizes.NumSampleTimes = 1;
sys = simsizes(sizes);
x0 = [0 0 0];
str = [];
ts = [0 0];
function sys = mdlDerivatives(t,x,u)
r = 1.5 * square(0.25 * t);
y = u(1);

load lqg_file; % Ac,Bcy,Bcw
sys(1:3) = Ac * x + Bcy * y + Bcw(:,2) * r;
function sys = mdlOutputs(t,x,u)
r = 1.5 * square(0.25 * t);
y = u(1);
load lqg_file; % Cc,Dcy,Dcw

xc = [x(1) x(2) x(3)]';
ut = Cc * xc + Dcy * y + Dcw(:,2) * r;

sys(1) = ut;
```

(4) 被控对象程序：chap5_1plant.m

```
function [sys,x0,str,ts] = s_function(t,x,u,flag)
switch flag,
case 0,
    [sys,x0,str,ts] = mdlInitializeSizes;
case 1,
    sys = mdlDerivatives(t,x,u);
```

```
case 3,
    sys = mdlOutputs(t,x,u);
case {2, 4, 9 }
    sys = [];
otherwise
    error(['Unhandled flag = ',num2str(flag)]);
end
function [sys,x0,str,ts] = mdlInitializeSizes
sizes = simsizes;
sizes.NumContStates   = 2;
sizes.NumDiscStates   = 0;
sizes.NumOutputs      = 1;
sizes.NumInputs       = 1;
sizes.DirFeedthrough  = 1;
sizes.NumSampleTimes  = 1;
sys = simsizes(sizes);
x0 = [0 0];
str = [];
ts = [0 0];
function sys = mdlDerivatives(t,x,u)
ut = u(1);
w0 = 0.001 * sin(10 * t);
Ap = [0 1; -1 -10];
Bpu = [0;1];
Bpw0 = Bpu;
sys(1:2) = Ap * x + Bpu * ut + Bpw0 * w0;
function sys = mdlOutputs(t,x,u)
ut = u(1);                      % sizes.DirFeedthrough = 1
Cpy = [1 0];
Dpyu = 0;
Dpyw0 = 1;
w0 = 0.001 * sin(10 * t);
v0 = 0.001 * rands(1);
sys = Cpy * x + Dpyu * ut + Dpyw0 * w0 + v0;
```

(5) 作图程序：chap5_1plot.m

```
close all;
r = 1.5 * square(0.25 * t);

figure(1);
plot(t,r,'-.r',t,y,'k','linewidth',2);
xlabel('time(s)');ylabel('x1 response');

figure(2);
plot(t,ut(:,1),'r','linewidth',2);
xlabel('time(s)');ylabel('Control input');
```

5.2 基于 LMI 的抗饱和闭环系统描述

5.2.1 系统描述

忽略噪声影响，仍考虑模型式(5.1)，即

$$\begin{aligned} \dot{\boldsymbol{x}}_{\mathrm{p}} &= \boldsymbol{A}_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{B}_{\mathrm{pu}}u + \boldsymbol{B}_{\mathrm{pw}}\boldsymbol{w} \\ y &= \boldsymbol{C}_{\mathrm{py}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pyu}}u + \boldsymbol{D}_{\mathrm{pyw}}\boldsymbol{w} \end{aligned} \tag{5.10}$$

控制的目标是通过施加一个控制输入 u，使 $y\to r$。由于输入受限部分的存在，闭环系统可能发散或者产生激烈的振荡，为保证系统稳定，需要设计抗饱和补偿器，控制输入受限控制系统如图 5.5 所示。

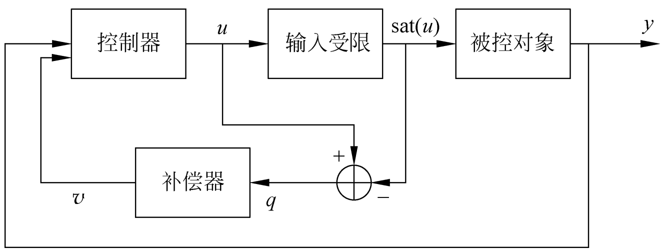


图 5.5 抗饱和补偿控制系统

控制输入饱和函数表达如下

$$\mathrm{sat}(u)=\begin{cases}\bar{u}, & u>\bar{u}\\ u, & -\bar{u}\leqslant u\leqslant\bar{u}\\ -\bar{u}, & u<\bar{u}\end{cases} \tag{5.11}$$

其中，u 为理论上的控制输入，取

$$q=u-\mathrm{sat}(u)$$

5.2.2 闭环系统的描述

参考文献[1]的设计方法，设计如下：

(1) 模型式(5.10)的位置状态方程表示

定义状态 $\boldsymbol{x}_{\mathrm{p}}$，输出为 y，性能输出函数为 z，则式(5.10)可写为

$$\begin{aligned} \dot{\boldsymbol{x}}_{\mathrm{p}} &= \boldsymbol{A}_{\mathrm{p}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{B}_{\mathrm{pu}}u + \boldsymbol{B}_{\mathrm{pw}}\boldsymbol{w} \\ y &= \boldsymbol{C}_{\mathrm{py}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pyu}}u + \boldsymbol{D}_{\mathrm{pyw}}\boldsymbol{w} \\ z &= \boldsymbol{C}_{\mathrm{pz}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pzu}}u + \boldsymbol{D}_{\mathrm{pzw}}\boldsymbol{w} \end{aligned} \tag{5.12}$$

(2) 设计抗饱和控制器

通过 LQG 控制器算法设计常规控制器式(5.9)，在此基础上，设计抗饱和控制器如下：

$$\begin{aligned}\dot{\boldsymbol{x}}_c &= \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_{cy}y + \boldsymbol{B}_{cw}\boldsymbol{w} + \boldsymbol{v}_1 \\ u &= \boldsymbol{C}_c\boldsymbol{x}_c + D_{cy}y + \boldsymbol{D}_{cw}\boldsymbol{w} + \boldsymbol{v}_2\end{aligned} \tag{5.13}$$

其中，$\boldsymbol{v}_1$ 和 $\boldsymbol{v}_2$ 为待求的补偿项。

(3) 输入受限时，不加补偿器 $\boldsymbol{v}$

此时式(5.12)可表示为

$$\begin{aligned}\dot{\boldsymbol{x}}_p &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_{pu}\mathrm{sat}(u) + \boldsymbol{B}_{pw}\boldsymbol{w} \\ y &= \boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{pyu}\mathrm{sat}(u) + \boldsymbol{D}_{pyw}\boldsymbol{w} \\ z &= \boldsymbol{C}_{pz}\boldsymbol{x}_p + \boldsymbol{D}_{pzu}\mathrm{sat}(u) + \boldsymbol{D}_{pzw}\boldsymbol{w}\end{aligned} \tag{5.14}$$

式(5.13)表示为

$$\begin{aligned}\dot{\boldsymbol{x}}_c &= \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_{cy}y + \boldsymbol{B}_{cw}\boldsymbol{w} \\ u &= \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_{cy}y + \boldsymbol{D}_{cw}\boldsymbol{w}\end{aligned} \tag{5.15}$$

由 $q = u - \mathrm{sat}(u)$，式(5.14)可写为

$$\begin{aligned}\dot{\boldsymbol{x}}_p &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_{pu}(u-q) + \boldsymbol{B}_{pw}\boldsymbol{w} \\ y &= \boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{pyu}(u-q) + \boldsymbol{D}_{pyw}\boldsymbol{w} \\ z &= \boldsymbol{C}_{pz}\boldsymbol{x}_p + \boldsymbol{D}_{pzu}(u-q) + \boldsymbol{D}_{pzw}\boldsymbol{w}\end{aligned} \tag{5.16}$$

将式(5.16)中的 y 代入式(5.15)，可得

$$u = \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_{cy}\left[\boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{pyu}(u-q) + \boldsymbol{D}_{pyw}\boldsymbol{w}\right] + \boldsymbol{D}_{cw}\boldsymbol{w}$$

即

$$u = \boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_{cy}\boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}(u-q) + \boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}\boldsymbol{w} + \boldsymbol{D}_{cw}\boldsymbol{w}$$

从而

$$u = (1 - \boldsymbol{D}_{cy}\boldsymbol{D}_{pyu})^{-1}\left[\boldsymbol{C}_c\boldsymbol{x}_c + \boldsymbol{D}_{cy}\boldsymbol{C}_{py}\boldsymbol{x}_p - \boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}q + (\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w}\right]$$

定义 $\boldsymbol{x}_{cl} = \begin{bmatrix}\boldsymbol{x}_p & \boldsymbol{x}_c\end{bmatrix}^{\mathrm{T}}$，$\Delta u = (1 - \boldsymbol{D}_{cy}\boldsymbol{D}_{pyu})^{-1}$，则 $\Delta u(1 - \boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}) = \boldsymbol{I}$，及

$$\boldsymbol{I} - \Delta u = -\Delta u\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}$$

从而可得

$$u = \Delta u\begin{bmatrix}\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{C}_c\end{bmatrix}\boldsymbol{x}_{cl} + (\boldsymbol{I} - \Delta u)q + \Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w} \tag{5.17}$$

将式(5.17)代入式(5.16)中的 $\dot{\boldsymbol{x}}_p$，可得

$$\begin{aligned}\dot{\boldsymbol{x}}_p &= \boldsymbol{A}_p\boldsymbol{x}_p + \boldsymbol{B}_{pu}\left[\Delta u\begin{bmatrix}\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{C}_c\end{bmatrix}\boldsymbol{x}_{cl} - \Delta uq + \Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w}\right] + \boldsymbol{B}_{pw}\boldsymbol{w} \\ &= \begin{bmatrix}\boldsymbol{A}_p + \boldsymbol{B}_{pu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{B}S_{pu}\Delta uC_c\end{bmatrix}\boldsymbol{x}_{cl} + \boldsymbol{B}_{pu}\left[-\Delta uq + \Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w}\right] + \boldsymbol{B}_{pw}\boldsymbol{w}\end{aligned} \tag{5.18}$$

将式(5.17)代入式(5.16)中的 y，可得

$$y = \boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{pyu}\left\{\Delta u\begin{bmatrix}\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{C}_c\end{bmatrix}\boldsymbol{x}_{cl} - \Delta uq + \Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w}\right\} + \boldsymbol{D}_{pyw}\boldsymbol{w} \tag{5.19}$$

将式(5.19)代入式(5.15)中的 $\dot{\boldsymbol{x}}_c$，可得

$$\begin{aligned}\dot{\boldsymbol{x}}_c = {} & \boldsymbol{A}_c\boldsymbol{x}_c + \boldsymbol{B}_{cy}\{\boldsymbol{C}_{py}\boldsymbol{x}_p + \boldsymbol{D}_{pyu}\left[\Delta u\begin{bmatrix}\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{C}_c\end{bmatrix}\boldsymbol{x}_{cl} - \Delta uq + \Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw} + \boldsymbol{D}_{cw})\boldsymbol{w}\right] + \\ & \boldsymbol{D}_{pyw}\boldsymbol{w}\} + \boldsymbol{B}_{cw}\boldsymbol{w} \\ = {} & \begin{bmatrix}\boldsymbol{B}_{cy}\boldsymbol{D}_{pyu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py} + \boldsymbol{B}_{cy}\boldsymbol{C}_{py} & \boldsymbol{A}_c + \boldsymbol{B}_{cy}\boldsymbol{D}_{pyu}\Delta u\boldsymbol{C}_c\end{bmatrix}\boldsymbol{x}_{cl} - \boldsymbol{B}_{cy}\boldsymbol{D}_{pyu}\Delta uq +\end{aligned}$$

$$[\boldsymbol{B}_{cy}\boldsymbol{D}_{pyu}\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{B}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{B}_{cw}]\boldsymbol{w} \tag{5.20}$$

由于$(1-\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy})\boldsymbol{D}_{pyu}=\boldsymbol{D}_{pyu}(1-\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu})$，则

$$\boldsymbol{D}_{pyu}(\boldsymbol{I}-\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu})^{-1}=(\boldsymbol{I}-\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy})^{-1}\boldsymbol{D}_{pyu}$$

定义 $\Delta y=(\boldsymbol{I}-\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy})^{-1}$，又由于 $\boldsymbol{\Delta u}=(\boldsymbol{I}-\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu})^{-1}$，则

$$\Delta y\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy}+\boldsymbol{I}=\Delta y \text{ 且 } \boldsymbol{D}_{pyu}\Delta u=\Delta y\boldsymbol{D}_{pyu}$$

则式(5.20)中的各项可整理为

$$\begin{aligned}
&\boldsymbol{B}_{cy}\boldsymbol{B}_{pyu}\Delta u\boldsymbol{B}_{cy}\boldsymbol{C}_{py}+\boldsymbol{B}_{cy}\boldsymbol{C}_{py}=\boldsymbol{B}_{cy}(\boldsymbol{B}_{pyu}\Delta u\boldsymbol{B}_{cy}+I)\boldsymbol{C}_{py}\\
&=\boldsymbol{B}_{cy}(\Delta y\boldsymbol{D}_{pyu}\boldsymbol{B}_{cy}+\boldsymbol{I})\boldsymbol{C}_{py}=\boldsymbol{B}_{cy}\Delta y\boldsymbol{C}_{py}-\boldsymbol{B}_{cy}\boldsymbol{D}_{pyu}\Delta u=-\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\\
&\boldsymbol{B}_{cy}\boldsymbol{B}_{pyu}\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{B}_{cy}\boldsymbol{B}_{pyw}+\boldsymbol{B}_{cw}\\
&=\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{B}_{cy}\boldsymbol{B}_{pyw}+\boldsymbol{B}_{cw}\\
&=\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{B}_{cy}\boldsymbol{B}_{pyw}+\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{D}_{cw}+\boldsymbol{B}_{cw}\\
&=\boldsymbol{B}_{cy}\Delta y(\boldsymbol{B}_{pyw}+\boldsymbol{D}_{pyu}\boldsymbol{D}_{cw})+\boldsymbol{B}_{cw}
\end{aligned}$$

其中，$\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{B}_{cy}\boldsymbol{B}_{pyw}=\boldsymbol{B}_{cy}(\Delta y\boldsymbol{D}_{pyu}\boldsymbol{D}_{cy}+\boldsymbol{I})\boldsymbol{D}_{pyw}=\boldsymbol{B}_{cy}\Delta y\boldsymbol{B}_{pyw}$。

则

$$\dot{\boldsymbol{x}}_c=[\boldsymbol{B}_{cy}\Delta y\boldsymbol{C}_{py}\boldsymbol{A}_c+\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{C}_c]\boldsymbol{x}_{cl}-\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}q+[\boldsymbol{B}_{cy}\Delta y(\boldsymbol{D}_{pyw}+\boldsymbol{D}_{pyu}\boldsymbol{D}_{cw})+\boldsymbol{B}_{cw}]\boldsymbol{w} \tag{5.21}$$

将式(5.17)代入式(5.16)，并根据 $\boldsymbol{I}-\Delta u=-\Delta u\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}$ 和式(5.19)，可得

$$\begin{aligned}
z&=C_{pz}\boldsymbol{x}_P+\boldsymbol{D}_{pzu}[\Delta u[\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{C}_c]\boldsymbol{x}_{cl}-\Delta u\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}q+\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})\boldsymbol{w}-q]+\boldsymbol{D}_{pzw}\boldsymbol{w}\\
&=[\boldsymbol{C}_{pz}+\boldsymbol{D}_{pzu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{D}_{pzu}\Delta u\boldsymbol{C}_c]\boldsymbol{x}_{cl}-\boldsymbol{D}_{pzu}(\Delta u\boldsymbol{D}_{cy}\boldsymbol{D}_{pyu}+\boldsymbol{I})q+\\
&=[\boldsymbol{D}_{pzu}\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{D}_{pzw}]\boldsymbol{w}\\
&=[\boldsymbol{C}_{pz}+\boldsymbol{D}_{pzu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{D}_{pzu}\Delta u\boldsymbol{C}_c]\boldsymbol{x}_{cl}-\boldsymbol{D}_{pzu}\Delta uq+\\
&\quad[\boldsymbol{D}_{pzu}\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{D}_{pzw}]\boldsymbol{w}
\end{aligned} \tag{5.22}$$

由式(5.17)、(5.18)、(5.21)和式(5.22)可得闭环系统的表示形式如下：

$$\begin{aligned}
u&=\Delta u[\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{C}_c]\boldsymbol{x}_{cl}+(\boldsymbol{I}-\Delta u)q+\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})\boldsymbol{w}\\
\dot{\boldsymbol{x}}_p&=[\boldsymbol{A}_p+\boldsymbol{B}_{pu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{B}_{pu}\Delta u\boldsymbol{C}_c]\boldsymbol{x}_{cl}+\boldsymbol{B}_{pu}[-\Delta uq+\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})\boldsymbol{w}]+\boldsymbol{B}_{pw}\boldsymbol{w}\\
\dot{\boldsymbol{x}}_c&=[\boldsymbol{B}_{cy}\Delta y\boldsymbol{C}_{py}\quad \boldsymbol{A}_c+\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{C}_c]\boldsymbol{x}_{cl}-\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}q+[\boldsymbol{B}_{cy}\Delta y(\boldsymbol{D}_{pyw}+\boldsymbol{D}_{pyu}\boldsymbol{D}_{cw})+\boldsymbol{B}_{cw}]\boldsymbol{w}\\
z&=[\boldsymbol{C}_{pz}+\boldsymbol{D}_{pzu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py}\quad \boldsymbol{D}_{pzu}\Delta u\boldsymbol{C}_c]\boldsymbol{x}_{cl}-\boldsymbol{D}_{pzu}\Delta uq+[\boldsymbol{D}_{pzu}\Delta u(\boldsymbol{D}_{cy}\boldsymbol{D}_{pyw}+\boldsymbol{D}_{cw})+\boldsymbol{D}_{pzw}]\boldsymbol{w}
\end{aligned}$$

从而可得闭环系统为

$$\begin{aligned}
\dot{\boldsymbol{x}}_{cl}&=\boldsymbol{A}_{cl}\boldsymbol{x}_{cl}+\boldsymbol{B}_{clq}q+\boldsymbol{B}_{clw}\boldsymbol{w}\\
\boldsymbol{z}&=\boldsymbol{C}_{clz}\boldsymbol{x}_{cl}+\boldsymbol{D}_{clzq}q+\boldsymbol{D}_{clzw}\boldsymbol{w}\\
u&=\boldsymbol{C}_{clu}\boldsymbol{x}_{cl}+\boldsymbol{D}_{cluq}q+\boldsymbol{D}_{cluw}\boldsymbol{w}
\end{aligned} \tag{5.23}$$

式(5.23)中采用了如下定义：

$$\boldsymbol{A}_{cl}=\begin{bmatrix}\boldsymbol{A}_p+\boldsymbol{B}_{pu}\Delta u\boldsymbol{D}_{cy}\boldsymbol{C}_{py} & \boldsymbol{B}_{pu}\Delta u\boldsymbol{C}_c\\ \boldsymbol{B}_{cy}\Delta y\boldsymbol{C}_{py} & \boldsymbol{A}_c+\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\boldsymbol{C}_c\end{bmatrix},\quad \boldsymbol{B}_{clq}=\begin{bmatrix}-\boldsymbol{B}_{pu}\Delta u\\ -\boldsymbol{B}_{cy}\Delta y\boldsymbol{D}_{pyu}\end{bmatrix},$$

$$\boldsymbol{B}_{\text{clw}}=\begin{bmatrix}\boldsymbol{B}_{\text{pw}}+\boldsymbol{B}_{\text{pu}}\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})\\ \boldsymbol{B}_{\text{cw}}+\boldsymbol{B}_{\text{cy}}\Delta y(\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}})\end{bmatrix},\quad \boldsymbol{C}_{\text{clz}}=[\boldsymbol{C}_{\text{pz}}+\boldsymbol{D}_{\text{pzu}}\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\quad \boldsymbol{D}_{\text{pzu}}\Delta u\boldsymbol{C}_{\text{c}}],$$

$\boldsymbol{D}_{\text{clzq}}=-\boldsymbol{D}_{\text{pzu}}\Delta u$， $\boldsymbol{D}_{\text{clzw}}=\boldsymbol{D}_{\text{pzu}}\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})+\boldsymbol{D}_{\text{pzw}}$， $\boldsymbol{C}_{\text{clu}}=\Delta u[\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\quad \boldsymbol{C}_{\text{c}}]$，

$\Delta\boldsymbol{D}_{\text{cluq}}=I-\Delta u$， $\boldsymbol{D}_{\text{cluw}}=\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})$， $\Delta y=(I-\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cy}})^{-1}$，

$\Delta u=(1-\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}})^{-1}$。

（4）输入受限时，加补偿器

此时式(5.10)可表示为

$$\begin{aligned}\dot{\boldsymbol{x}}_{\text{p}}&=\boldsymbol{A}_{\text{p}}\boldsymbol{x}_{\text{p}}+\boldsymbol{B}_{\text{pu}}u+\boldsymbol{B}_{\text{pw}}\boldsymbol{w}\\ y&=\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}u+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}\\ z&=\boldsymbol{C}_{\text{pz}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pzu}}u+\boldsymbol{D}_{\text{pzw}}\boldsymbol{w}\end{aligned}\tag{5.24}$$

式(5.15)表示为

$$\begin{aligned}\dot{\boldsymbol{x}}_{\text{c}}&=\boldsymbol{A}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{B}_{\text{cy}}y+\boldsymbol{B}_{\text{cw}}\boldsymbol{w}+v_1\\ u&=\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{cy}}y+\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+v_2\end{aligned}\tag{5.25}$$

由 $q=u-\text{sat}(u)$，式(5.13)可写为

$$\begin{aligned}\dot{\boldsymbol{x}}_{\text{p}}&=\boldsymbol{A}_{\text{p}}\boldsymbol{x}_{\text{p}}+\boldsymbol{B}_{\text{pu}}(u-q)+\boldsymbol{B}_{\text{pw}}\boldsymbol{w}\\ y&=\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}(u-q)+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}\\ z&=\boldsymbol{C}_{\text{pz}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pzu}}(u-q)+\boldsymbol{D}_{\text{pzw}}\boldsymbol{w}\end{aligned}\tag{5.26}$$

将式(5.26)中的 $y=\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}(u-q)+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}$ 代入式(15)，可得

$$u=\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{cy}}(\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}(u-q)+\boldsymbol{D}_{\text{pyw}})\boldsymbol{w}+\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+[0\quad \boldsymbol{I}]\boldsymbol{D}_{\text{aw}}q$$

即

$$u=\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}u-\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}q+\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}+\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+[0\quad \boldsymbol{I}]\boldsymbol{D}_{\text{aw}}q$$

由 $\Delta u=(1-\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}})^{-1}$ 可知 $-\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}=\boldsymbol{I}-\Delta u$，从而

$$\begin{aligned}u&=(1-\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}})^{-1}\{\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}-\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}q+\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}+\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+[0\quad \boldsymbol{I}]\boldsymbol{D}_{\text{aw}}q\}\\ &=[\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\quad \Delta u\boldsymbol{C}_{\text{c}}]\boldsymbol{x}_{\text{cl}}+([0\quad \Delta u]\boldsymbol{D}_{\text{aw}}-\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}})q+\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})\boldsymbol{w}\\ &=\boldsymbol{C}_{\text{clu}}\boldsymbol{x}_{\text{cl}}+(\boldsymbol{D}_{\text{cluq}}+\boldsymbol{D}_{\text{cluv}}\boldsymbol{D}_{\text{aw}})q+\boldsymbol{D}_{\text{cluw}}\boldsymbol{w}\end{aligned}\tag{5.27}$$

其中，$\boldsymbol{D}_{\text{cluq}}=-\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}$，$\boldsymbol{D}_{\text{cluq}}=-\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}=I-\Delta u$，$\boldsymbol{D}_{\text{cluv}}=[0\quad \Delta u]$。

将式(5.27)代入式(5.26)中的 $\dot{\boldsymbol{x}}_{\text{p}}$，可得

$$\begin{aligned}\dot{\boldsymbol{x}}_{\text{p}}&=\boldsymbol{A}_{\text{p}}\boldsymbol{x}_{\text{p}}+\boldsymbol{B}_{\text{pu}}(\boldsymbol{C}_{\text{clu}}\boldsymbol{x}_{\text{cl}}+(\boldsymbol{D}_{\text{cluq}}+\boldsymbol{D}_{\text{cluv}}\boldsymbol{D}_{\text{aw}})q+\boldsymbol{D}_{\text{cluw}}\boldsymbol{w}-q)+\boldsymbol{B}_{\text{pw}}\boldsymbol{w}\\ &=\boldsymbol{A}_{\text{p}}\boldsymbol{x}_{\text{p}}-\boldsymbol{B}_{\text{pu}}q+\boldsymbol{B}_{\text{pu}}\{[\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\quad \Delta u\boldsymbol{C}_{\text{c}}]\boldsymbol{x}_{\text{cl}}+(\boldsymbol{D}_{\text{cluq}}+\boldsymbol{D}_{\text{cluv}}\boldsymbol{D}_{\text{aw}})q+\boldsymbol{D}_{\text{cluw}}\boldsymbol{w}\}+\boldsymbol{B}_{\text{pw}}\boldsymbol{w}\\ &=[\boldsymbol{A}_{\text{p}}+\boldsymbol{B}_{\text{pu}}\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}}\quad \boldsymbol{B}_{\text{pu}}\Delta u\boldsymbol{C}_{\text{c}}]\boldsymbol{x}_{\text{cl}}+(-\boldsymbol{B}_{\text{pu}}\Delta u+\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}\boldsymbol{D}_{\text{aw}})q+\\ &\qquad[\boldsymbol{B}_{\text{pu}}\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})+\boldsymbol{B}_{\text{pw}}]\boldsymbol{w}\end{aligned}\tag{5.28}$$

其中，$-\boldsymbol{B}_{\text{pu}}q+\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluq}}q=(-\boldsymbol{B}_{\text{pu}}+\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluq}})q=[-\boldsymbol{B}_{\text{pu}}+\boldsymbol{B}_{\text{pu}}(I-\Delta u)]q=-\boldsymbol{B}_{\text{pu}}\Delta uq$。

将 u 代入 y

$$y=\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+(\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cy}}y+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}}w+\boldsymbol{D}_{\text{pyu}}v_2-\boldsymbol{D}_{\text{pyu}}q)+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}$$

则

$$y=(\boldsymbol{I}-\boldsymbol{D}_{\text{pyu}}D_{\text{cy}})^{-1}(\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+\boldsymbol{D}_{\text{pyu}}v_2-\boldsymbol{D}_{\text{pyu}}q+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w})$$
$$=\Delta y(\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+\boldsymbol{D}_{\text{pyu}}v_2-\boldsymbol{D}_{\text{pyu}}q+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w})$$

$$\begin{aligned}\dot{\boldsymbol{x}}_{\text{c}}&=\boldsymbol{A}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{B}_{\text{cy}}y+\boldsymbol{B}_{cw}\boldsymbol{w}+v_1\\
&=\boldsymbol{A}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y(\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+\boldsymbol{D}_{\text{pyu}}v_2-\boldsymbol{D}_{\text{pyu}}\boldsymbol{q}+\boldsymbol{D}_{\text{pyw}}\boldsymbol{w})+\boldsymbol{B}_{\text{cw}}\boldsymbol{w}+v_1\\
&=\boldsymbol{A}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{C}_{\text{py}}\boldsymbol{x}_{\text{p}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\boldsymbol{x}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}}\boldsymbol{w}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}v_2-\\
&\quad\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{q}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyw}}\boldsymbol{w}+\boldsymbol{B}_{\text{cw}}\boldsymbol{w}+v_1\\
&=[\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{C}_{\text{py}}\quad \boldsymbol{A}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}]\boldsymbol{x}_{\text{cl}}-\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{q}+(v_1+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}v_2)+\\
&\quad\boldsymbol{B}_{\text{cy}}\Delta y(\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}})\boldsymbol{w}+\boldsymbol{B}_{\text{cw}}\boldsymbol{w}\\
&=[\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{C}_{\text{py}}\quad \boldsymbol{A}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}]\boldsymbol{x}_{\text{cl}}+\{-\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}+[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]\boldsymbol{D}_{\text{aw}}\}q+\\
&\quad\boldsymbol{B}_{\text{cy}}\Delta y(\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}})\boldsymbol{w}+\boldsymbol{B}_{\text{cw}}\boldsymbol{w}\end{aligned}$$

其中，$v_1+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}v_2=[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]\boldsymbol{v}=[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]\boldsymbol{D}_{\text{aw}}q$。

将 $\dot{\boldsymbol{x}}_{\text{p}}$ 与 $\dot{\boldsymbol{x}}_{\text{c}}$ 合并，可得

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{x}}_{\text{p}}\\ \dot{\boldsymbol{x}}_{\text{c}}\end{bmatrix}&=\begin{bmatrix}\boldsymbol{A}_{\text{p}}+\boldsymbol{B}_{\text{pu}}\Delta u\boldsymbol{D}_{\text{cy}}\boldsymbol{C}_{\text{py}} & \boldsymbol{B}_{\text{pu}}\Delta u\boldsymbol{C}_{\text{c}}\\ \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{C}_{\text{py}} & \boldsymbol{A}_{\text{c}}+\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\boldsymbol{C}_{\text{c}}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_{\text{p}}\\ \boldsymbol{x}_{\text{c}}\end{bmatrix}+\\
&\quad\begin{bmatrix}-\boldsymbol{B}_{\text{pu}}\Delta u\\ -\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}q+\begin{bmatrix}0 & \boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}\\ \boldsymbol{I} & \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}\boldsymbol{D}_{\text{aw}}q+\\
&\quad\begin{bmatrix}\boldsymbol{B}_{\text{pu}}\Delta u(\boldsymbol{D}_{\text{cy}}\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{cw}})+\boldsymbol{B}_{\text{pw}}\\ \boldsymbol{B}_{\text{cy}}\Delta y(\boldsymbol{D}_{\text{pyw}}+\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cw}})+\boldsymbol{B}_{\text{cw}}\end{bmatrix}\boldsymbol{w}\\
&=\boldsymbol{A}_{\text{cl}}\boldsymbol{x}_{\text{cl}}+\boldsymbol{B}_{\text{clq}}q+\boldsymbol{B}_{\text{clv}}\boldsymbol{D}_{\text{aw}}q+\boldsymbol{B}_{\text{clw}}\boldsymbol{w}\end{aligned}$$

其中，

$$\begin{aligned}\begin{bmatrix}-\boldsymbol{B}_{\text{pu}}\Delta u+\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}\boldsymbol{D}_{\text{aw}}\\ -\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}+[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]\boldsymbol{D}_{\text{aw}}\end{bmatrix}q&=\begin{bmatrix}-\boldsymbol{B}_{\text{pu}}\Delta u\\ -\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}q+\begin{bmatrix}\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}\\ [\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]\end{bmatrix}\boldsymbol{D}_{\text{aw}}q\\
&=\begin{bmatrix}-\boldsymbol{B}_{\text{pu}}\Delta u\\ -\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}q+\begin{bmatrix}0 & \boldsymbol{B}_{\text{pu}}\Delta u\\ \boldsymbol{I} & \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}\boldsymbol{D}_{\text{aw}}q\\
&=\boldsymbol{B}_{\text{clq}}q+\boldsymbol{B}_{\text{clv}}\boldsymbol{D}_{\text{aw}}q\end{aligned}$$

即

$$\dot{\boldsymbol{x}}_{\text{cl}}=\boldsymbol{A}_{\text{cl}}\boldsymbol{x}_{\text{cl}}+(\boldsymbol{B}_{\text{clq}}+\boldsymbol{B}_{\text{clv}}\boldsymbol{D}_{\text{aw}})q+\boldsymbol{B}_{\text{clw}}\boldsymbol{w}$$

其中，$\boldsymbol{B}_{\text{clq}}=\begin{bmatrix}-\boldsymbol{B}_{\text{pu}}\Delta u\\ -\boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}$，$\boldsymbol{B}_{\text{clv}}=\begin{bmatrix}\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}\\ \boldsymbol{B}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cluv}}+[\boldsymbol{I}\quad 0]\end{bmatrix}=\begin{bmatrix}0 & \boldsymbol{B}_{\text{pu}}\Delta u\\ \boldsymbol{I} & \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}\end{bmatrix}$，$\boldsymbol{B}_{\text{pu}}\boldsymbol{D}_{\text{cluv}}=\boldsymbol{B}_{\text{pu}}[0\quad \Delta u]=[0\quad \boldsymbol{B}_{\text{pu}}\Delta u]$，$\boldsymbol{B}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}\boldsymbol{D}_{\text{cluv}}+[\boldsymbol{I}\quad 0]=\boldsymbol{B}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}[0\quad \Delta u]+[\boldsymbol{I}\quad 0]=[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\boldsymbol{D}_{\text{pyu}}\Delta u]=[\boldsymbol{I}\quad \boldsymbol{B}_{\text{cy}}\Delta y\boldsymbol{D}_{\text{pyu}}]$。

将 u 代入 z 中，并代入 $\boldsymbol{C}_{\text{clu}}$、$\boldsymbol{D}_{\text{cluq}}$、$\boldsymbol{D}_{\text{cluw}}$，可得

$$
\begin{aligned}
z &= \boldsymbol{C}_{\mathrm{pz}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pzu}}u - \boldsymbol{D}_{\mathrm{pzu}}q + \boldsymbol{D}_{\mathrm{pzw}}\boldsymbol{w} \\
&= \boldsymbol{C}_{\mathrm{pz}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pzu}}[\boldsymbol{C}_{\mathrm{clu}}\boldsymbol{x}_{\mathrm{cl}} + (\boldsymbol{D}_{\mathrm{cluq}} + \boldsymbol{D}_{\mathrm{cluv}}\boldsymbol{D}_{\mathrm{aw}})q + \boldsymbol{D}_{\mathrm{cluw}}\boldsymbol{w}] - \boldsymbol{D}_{\mathrm{pzu}}q + \boldsymbol{D}_{\mathrm{pzw}}\boldsymbol{w} \\
&= \boldsymbol{C}_{\mathrm{pz}}\boldsymbol{x}_{\mathrm{p}} + \boldsymbol{D}_{\mathrm{pzu}}\{[\Delta u\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{C}_{\mathrm{py}} \quad \Delta u\boldsymbol{C}_{\mathrm{c}}]\boldsymbol{x}_{\mathrm{cl}} + \{[I - \Delta u] + \boldsymbol{D}_{\mathrm{cluv}}\boldsymbol{D}_{\mathrm{aw}}\}q + \Delta u(\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{D}_{\mathrm{pyw}} + \\
&\quad \boldsymbol{D}_{\mathrm{cw}})\boldsymbol{w}\} - \boldsymbol{D}_{\mathrm{pzu}}q + \boldsymbol{D}_{\mathrm{pzw}}\boldsymbol{w} \\
&= [\boldsymbol{C}_{\mathrm{pz}} + \boldsymbol{D}_{\mathrm{pzu}}\Delta u\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{C}_{\mathrm{py}} \quad \boldsymbol{D}_{\mathrm{pzu}}\Delta u\boldsymbol{C}_{\mathrm{c}}]\boldsymbol{x}_{\mathrm{cl}} + (-\boldsymbol{D}_{\mathrm{pzu}}\Delta u + \boldsymbol{D}_{\mathrm{pzu}}\boldsymbol{D}_{\mathrm{cluv}}\boldsymbol{D}_{\mathrm{aw}})q + \\
&\quad [\boldsymbol{D}_{\mathrm{pzu}}\Delta u(\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{D}_{\mathrm{pyw}} + \boldsymbol{D}_{\mathrm{cw}}) + \boldsymbol{D}_{\mathrm{pzw}}]\boldsymbol{w} \\
&= [\boldsymbol{C}_{\mathrm{pz}} + \boldsymbol{D}_{\mathrm{pzu}}\Delta u\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{C}_{\mathrm{py}} \quad \boldsymbol{D}_{\mathrm{pzu}}\Delta u\boldsymbol{C}_{\mathrm{c}}]\boldsymbol{x}_{\mathrm{cl}} + \{-\boldsymbol{D}_{\mathrm{pzu}}\Delta u + [0 \quad \boldsymbol{D}_{\mathrm{pzu}}\Delta u]\boldsymbol{D}_{\mathrm{aw}}\}q + \\
&\quad [\boldsymbol{D}_{\mathrm{pzu}}\Delta u(\boldsymbol{D}_{\mathrm{cy}}\boldsymbol{D}_{\mathrm{pyw}} + \boldsymbol{D}_{\mathrm{cw}}) + \boldsymbol{D}_{\mathrm{pzw}}]\boldsymbol{w} \\
&= \boldsymbol{C}_{\mathrm{clz}}\boldsymbol{x}_{\mathrm{cl}} + (\boldsymbol{D}_{\mathrm{clzq}} + \boldsymbol{D}_{\mathrm{clzv}}\boldsymbol{D}_{\mathrm{aw}})q + \boldsymbol{D}_{\mathrm{clzw}}\boldsymbol{w}
\end{aligned}
$$

其中 $\boldsymbol{D}_{\mathrm{pzu}}\boldsymbol{D}_{\mathrm{cluv}}\boldsymbol{D}_{\mathrm{aw}} = \boldsymbol{D}_{\mathrm{pzu}}[0 \quad \Delta u]\boldsymbol{D}_{\mathrm{aw}} = [0 \quad \boldsymbol{D}_{\mathrm{pzu}}\Delta u]\boldsymbol{D}_{\mathrm{aw}}$。

$$
\boldsymbol{D}_{\mathrm{clzv}} = \boldsymbol{D}_{\mathrm{pzu}}\boldsymbol{D}_{\mathrm{cluv}} = \boldsymbol{D}_{\mathrm{pzu}}[0 \quad \Delta u] = [0 \quad \boldsymbol{D}_{\mathrm{pzu}}\Delta u]
$$

从而可得闭环系统

$$
\begin{aligned}
\dot{\boldsymbol{x}}_{\mathrm{cl}} &= \boldsymbol{A}_{\mathrm{cl}}\boldsymbol{x}_{\mathrm{cl}} + (\boldsymbol{B}_{\mathrm{clv}}\boldsymbol{D}_{\mathrm{aw}} + \boldsymbol{B}_{\mathrm{clq}})q + \boldsymbol{B}_{\mathrm{clw}}\boldsymbol{w} \\
z &= \boldsymbol{C}_{\mathrm{clz}}\boldsymbol{x}_{\mathrm{cl}} + (\boldsymbol{D}_{\mathrm{clzq}} + \boldsymbol{D}_{\mathrm{clzv}}\boldsymbol{D}_{\mathrm{aw}})q + \boldsymbol{D}_{\mathrm{clzw}}\boldsymbol{w} \\
u &= \boldsymbol{C}_{\mathrm{clu}}\boldsymbol{x}_{\mathrm{cl}} + (\boldsymbol{D}_{\mathrm{cluq}} + \boldsymbol{D}_{\mathrm{cluv}}\boldsymbol{D}_{\mathrm{aw}})q + \boldsymbol{D}_{\mathrm{cluw}}\boldsymbol{w}
\end{aligned}
\tag{5.29}
$$

其中，$\boldsymbol{B}_{\mathrm{clv}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B}_{\mathrm{pu}}\Delta u \\ \boldsymbol{I} & \boldsymbol{B}_{\mathrm{cy}}\Delta y\boldsymbol{D}_{\mathrm{pyu}} \end{bmatrix}$，$\boldsymbol{D}_{\mathrm{cluv}} = [\boldsymbol{0} \quad \Delta u]$。

(5) 控制器设计

考虑被控对象模型为

$$
\frac{1}{s^2 + 10s + 1}
$$

上式写为状态方程形式

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -10x_2 - x_1 + u(t) \\
y &= x_1 \\
z &= y - r
\end{aligned}
$$

对应于模型式(5.12)的信息为

$$
\boldsymbol{A}_{\mathrm{p}} = \begin{bmatrix} 0 & 1 \\ -1 & -10 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{pu}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{pw}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{C}_{\mathrm{py}} = [1 \quad 0],
$$

$$
\boldsymbol{D}_{\mathrm{pyu}} = 0, \quad \boldsymbol{D}_{\mathrm{pyw}} = 0, \quad \boldsymbol{C}_{\mathrm{pz}} = \boldsymbol{C}_{\mathrm{py}}, \quad \boldsymbol{D}_{\mathrm{pzu}} = \boldsymbol{D}_{\mathrm{pyu}}, \quad \boldsymbol{D}_{\mathrm{pzw}} = [0 \quad -1]。
$$

采用 LQG 最优控制，可得对应于控制器(5.13)的信息为

$$
\boldsymbol{A}_{\mathrm{c}} = \begin{bmatrix} -0.0081 & 1 & 0 \\ -203.8428 & -74.1936 & 244.9490 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_{\mathrm{cy}} = [0.0081 \quad 0.50 \quad -1.0]^{\mathrm{T}},
$$

$$
\boldsymbol{B}_{\mathrm{cw}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{C}_{\mathrm{c}} = [-202.3428 \quad -64.1936 \quad 244.9490], \quad \boldsymbol{D}_{\mathrm{cy}} = 0, \boldsymbol{D}_{\mathrm{cw}} = [0 \quad 0]。
$$

控制器(5.13)中的$\boldsymbol{v}_1$和$\boldsymbol{v}_2$由下面补偿算法求得。

5.2.3 补偿算法设计步骤

补偿项$\boldsymbol{D}_{aw}$的求法分以下几步[1,2]：

第一步：求解最小γ和$\boldsymbol{R}$

开环系统：

$$\begin{bmatrix} R_{11}\boldsymbol{A}_p^T+\boldsymbol{A}_pR_{11} & \boldsymbol{B}_{pw} & R_{11}\boldsymbol{C}_{pz}^T \\ \boldsymbol{B}_{pw}^T & -\gamma\boldsymbol{I}_{nw} & \boldsymbol{D}_{pzw}^T \\ \boldsymbol{C}_{pz}R_{11} & \boldsymbol{D}_{pzw} & -\gamma\boldsymbol{I}_{nz} \end{bmatrix}<0 \tag{5.30}$$

闭环系统：

$$\begin{bmatrix} \boldsymbol{R}\boldsymbol{A}_{cl}^T+\boldsymbol{A}_{cl}\boldsymbol{R} & \boldsymbol{B}_{clw} & \boldsymbol{R}\boldsymbol{C}_{clz}^T \\ \boldsymbol{B}_{clw}^T & -\gamma\boldsymbol{I}_{nw} & \boldsymbol{D}_{clzw}^T \\ \boldsymbol{C}_{clz}R & \boldsymbol{D}_{clzw} & -\gamma\boldsymbol{I}_{nz} \end{bmatrix}<0 \tag{5.31}$$

$$\boldsymbol{R}=\boldsymbol{R}^T>0 \tag{5.32}$$

上面条件下，求使满足$\min\limits_{\boldsymbol{R},\gamma}\gamma$的$\gamma$和$\boldsymbol{R}$。

第二步：求解$\boldsymbol{D}_{aw}$

基于抗饱和增益求解的LMI式为

$$\boldsymbol{\Psi}(\boldsymbol{U},\gamma)+\boldsymbol{G}_U^T\boldsymbol{\Lambda}_U^T\boldsymbol{H}^T+\boldsymbol{H}\boldsymbol{\Lambda}_U\boldsymbol{G}_U<0 \tag{5.33}$$

其中，

$$\boldsymbol{\Psi}(\boldsymbol{U},\gamma)=\mathrm{He}\begin{bmatrix} \boldsymbol{A}_{cl}\boldsymbol{R} & \boldsymbol{B}_{clq}\boldsymbol{U}+\boldsymbol{Q}\boldsymbol{C}_{clu}^T & \boldsymbol{B}_{clw} & \boldsymbol{Q}\boldsymbol{C}_{clz}^T \\ 0 & \boldsymbol{D}_{cluq}\boldsymbol{U}-\boldsymbol{U} & \boldsymbol{D}_{clyw} & \boldsymbol{U}\boldsymbol{D}_{clzq}^T \\ 0 & 0 & -\dfrac{\gamma}{2}\boldsymbol{I} & \boldsymbol{D}_{clzw}^T \\ 0 & 0 & 0 & -\dfrac{\gamma}{2}\boldsymbol{I} \end{bmatrix}$$

$$\boldsymbol{H}=[\boldsymbol{B}_{clv}^T \quad \boldsymbol{0} \quad | \quad \boldsymbol{D}_{cluv}^T \quad | \quad \boldsymbol{0} \quad | \quad \boldsymbol{D}_{clzv}^T]$$

$$\boldsymbol{G}_U=[\boldsymbol{0} \quad \boldsymbol{0} \quad | \quad \boldsymbol{I} \quad | \quad \boldsymbol{0} \quad \boldsymbol{0}]$$

其中，$\mathrm{He}(\boldsymbol{X})=\boldsymbol{X}+\boldsymbol{X}^T$。

由于采用的是静态补偿器，故$n_{aw}=0$，$\boldsymbol{Q}=\boldsymbol{R}$。

为了保证闭环系统的强稳定性，$\boldsymbol{\Lambda}_U$和$\boldsymbol{U}$需要满足如下LMI不等式[1]：

$$-2(1-\mu)\boldsymbol{U}+\mathrm{He}(\boldsymbol{D}_{cluq}\boldsymbol{U}+[\boldsymbol{0}_{nu\times nc} \quad \boldsymbol{I}_{nu}]\boldsymbol{\Lambda}_U)<0 \tag{5.34}$$

其中，μ是很小的正实数。

求解不等式(5.30)至式(5.34)，可得$\boldsymbol{\Lambda}_U$和$\boldsymbol{U}$，从而可得静态补偿器为

$$\boldsymbol{D}_{aw}=\boldsymbol{\Lambda}_U\boldsymbol{U}^{-1}, \quad v=\boldsymbol{D}_{aw}q \tag{5.35}$$

5.2.4 闭环系统稳定性分析

首先，进行闭环系统稳定性分析，设计 Lyapunov 函数 $\boldsymbol{V}=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$，通过取 $\dot{V}<0$ 并进行变换，可得到式(5.33)。

然后，根据文献[2]中的引理 5 可知，式(5.33)有解的条件是当且仅当文献[2]中式(31)成立，依据是矩阵消除定理和[2]中定理 2、定理 5 的证明，从而可得到式(5.30)至式(5.32)。

式(5.30)至式(5.33)的证明和分析来源于文献[2]。

5.2.5 仿真实例

考虑被控对象为式(5.10)，初始状态为 [0 0 0]，指令为方波信号，采用 MATLAB 函数实现，取指令为 $r=1.5\text{square}(0.08t)$。求解不等式式(5.30)至式(5.34)，取 $v_0=0$，可得

$$\gamma=2.7518,\quad \boldsymbol{D}_{\mathrm{aw}}=[-0.0002\quad 0.0019\quad 0.0006\quad 1.0]$$

按式(5.25)设计控制律，控制输入受限值为 $\bar{u}=5.0$，如果加补偿器，根据式(5.35)求补偿 $v=\boldsymbol{D}_{\mathrm{aw}}q$，仿真结果如图 5.6 和图 5.7 所示。如果不加补偿器，取 $\boldsymbol{D}_{\mathrm{aw}}=0$，$\boldsymbol{v}=0$，仿真结果如图 5.8 和图 5.9 所示。可见，加入补偿器后，方波跟踪性能和控制输入信号得到了很大的改善。

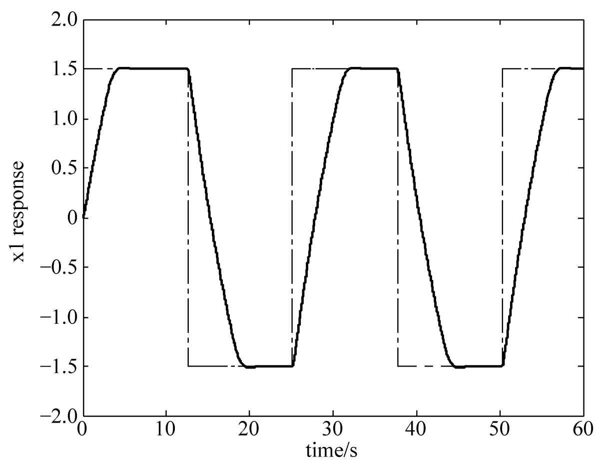


图 5.6 方波响应(加补偿器)

仿真程序：

(1) LMI 设计程序：chap5_2LMI. m

```
% Generic antiwindup LMI program
clear all;
close all;
% Dimension defination
np = 2;                                    % 被控对象动态阶数
```

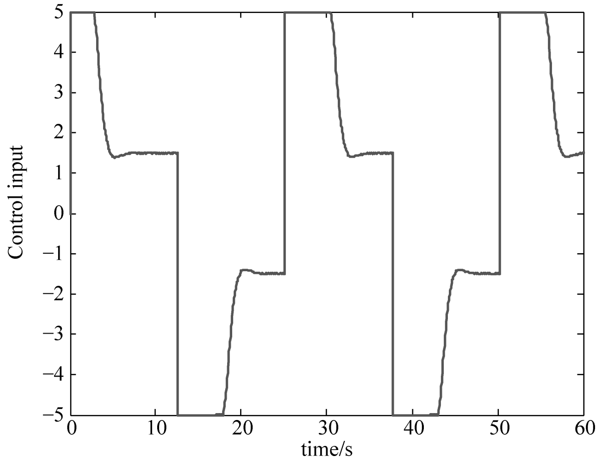


图 5.7　控制输入(加补偿器)

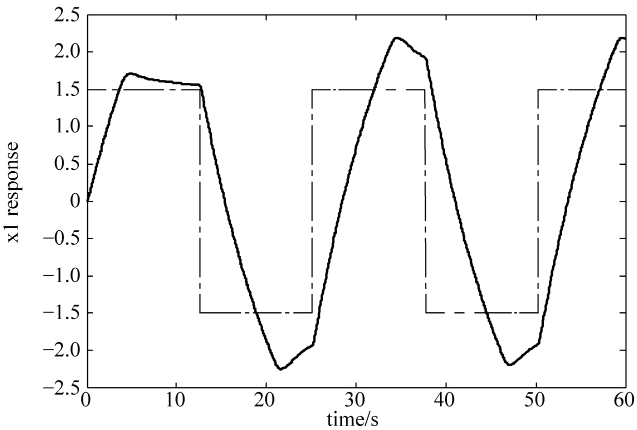


图 5.8　方波响应(不加补偿器)

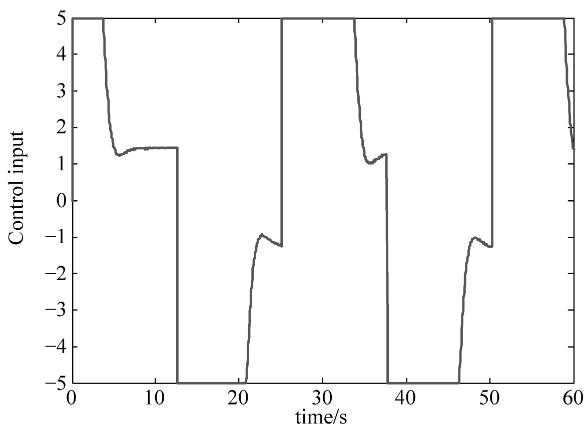


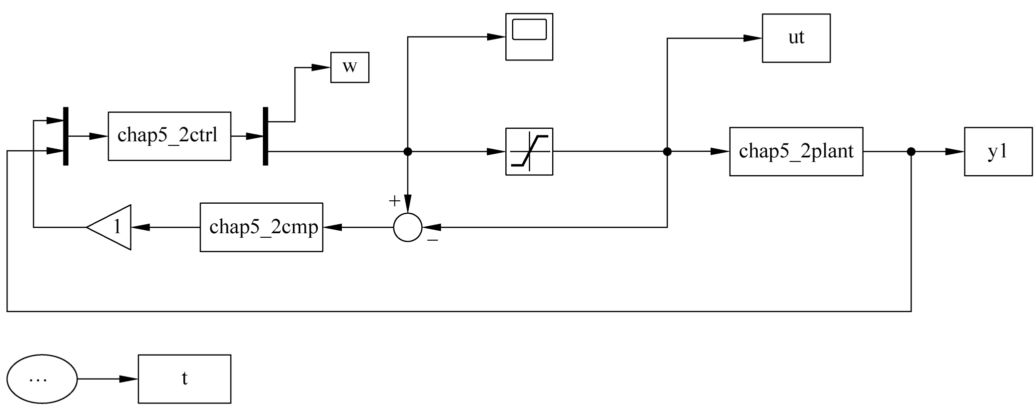
图 5.9　控制输入(不加补偿器)

```
nc = 3;                                          % 控制器动态阶数
nu = 1;                                          % 控制输入个数
nw = 2;                                          % 外加输入个数
nz = 1;                                          % 性能指标个数
naw = 0;                                         % 补偿器状态阶数
ncl = np + nc;
nv = nu + nc;
npc = np + nc;
n = np + nc + naw;
m = ncl + nu + nw + nz;
% Plant defination
Ap = [0 1; -1 -10];
Bpu = [0;1];
Cpy = [1 0];
Dpyu = 0;

Bpw0 = Bpu;
Bpr = zeros(2,1);
Bpw = [Bpw0 Bpr];

Dpyw0 = 1;
Dpyr = 0;
Dpyw = [Dpyw0 Dpyr];
% error performance
Cpz = Cpy;
Dpzu = Dpyu;
Dpzw = Dpyw - [0 1];

% other performance
nx = 2;                                          %状态维数
ny = 1;                                          %输出维数
sys = ss(Ap,Bpu,Cpy,Dpyu);

%Used in lqi
Q = [5 0 0;0 5 0;0 0 60];
R1 = 0.001;
N = 0;
K = lqi(sys,Q,R1,N);
Kx = K(1:nx);
Ki = K(nx + 1:nx + ny);

Plant = ss(Ap,[Bpu Bpw0],Cpy,[Dpyu Dpyw0]);
Qn = 10;Rn = 10;Nn = 0;
[kalmf,L] = kalman(Plant,Qn,Rn,Nn);

Ac = [Ap - Bpu * Kx - L * Cpy + L * Dpyu * Kx  - Bpu * Ki + L * Dpyu * Ki;
    zeros(ny,nx) zeros(ny,ny)];
Bcy = [L; - eye(ny)];
Bcwr = [zeros(nx,1);eye(ny)];
Bcw = [zeros(3,1) Bcwr];
```

```
Cc = - K;
Dcy = 0;
Dcw = [0 0];

deltau = inv(eye(1) - Dcy * Dpyu);
deltay = inv(eye(1) - Dpyu * Dcy);

Acl = [Ap + Bpu * deltau * Dcy * Cpy Bpu * deltau * Cc;
    Bcy * deltay * Cpy Ac + Bcy * deltay * Dpyu * Cc];
Cclz = [Cpz + Dpzu * deltau * Dcy * Cpy Dpzu * deltau * Cc];
Cclu = [deltau * Dcy * Cpy deltau * Cc];
Bclq = [ - Bpu * deltau; - Bcy * deltay * Dpyu];
Bclw = [Bpw + Bpu * deltau * (Dcw + Dcy * Dpyw);Bcw + Bcy * deltay * (Dpyw + Dpyu * Dcw)];

Dclzq = - (Dpzu * deltau);
Dclzw = Dpzw + Dpzu * deltau * (Dcw + Dcy * Dpyw);

Dcluw = deltau * (Dcw + Dcy * Dpyw);
Dcluq = eye(1) - deltau;

Bclv = [zeros(np,nc) Bpu * deltau;
      eye(nc) Bcy * deltay * Dpyu];
Dclzv = [zeros(nz,nc) Dpzu * deltau];
Dcluv = [zeros(nu,nc) deltau];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  ga = sdpvar(1);
  R11 = sdpvar(np);
  R12 = sdpvar(np,nc);
  R22 = sdpvar(nc);
  R = [R11 R12;R12' R22];

  Openloop = [R11 * Ap' + Ap * R11 Bpw R11 * Cpz';Bpw' - ga * eye(nw) Dpzw';Cpz * R11 Dpzw - ga *
eye(nz)];
  Closedloop = [R * Acl' + Acl * R Bclw R * Cclz';Bclw' - ga * eye(nw) Dclzw';Cclz * R Dclzw - ga
 * eye(nz)];

  F = set(Openloop < 0) + set(Closedloop < 0) + set(R > 0);  % First step, LMI to get R and gama
  solvesdp(F,ga);
  gama = double(ga);

   R = double(R)
   Q = R;
   U = sdpvar(nu);
   Au = sdpvar(naw + nv,naw + nu);
   ga = sdpvar(1);

   fai = [(Acl * R) + (Acl * R)' Bclq * U + Q * Cclu' Bclw Q * Cclz';
       (Bclq * U + Q * Cclu')' Dcluq * U - U + (Dcluq * U - U)' Dcluw U * Dclzq';
       Bclw' Dcluw' - ga * eye(nw) Dclzw';
       (Q * Cclz')' (U * Dclzq')' Dclzw  - ga * eye(nz)];
```

```
% H1 = [Bclv' zeros(naw,nv)]; % naw = 0
  H1 = Bclv';
  H2 = Dcluv';
  H3 = Dclzv';
  H = [H1 H2 zeros(naw + nv,nw) H3];
% GU = [zeros(naw,nu + nw + nz);zeros(nu,n) eye(nu) zeros(nu,nw + nz)]; % naw = 0
  GU = [zeros(nu,n) eye(nu) zeros(nu,nw + nz)];

  Mu = 0;
  Stro = - 2 * (1 - Mu) * U + (Dcluq * U + [zeros(nu,nc) eye(nu)] * Au) + (Dcluq * U +
[zeros(nu,nc) eye(nu)] * Au)';
  Anti = fai + GU' * Au' * H + H' * Au * GU;

  Fa = set(Stro < 0) + set(Anti < 0);            % Second step, LMI to get Au and UU
   solvesdp(Fa,ga);
   gama = double(ga)
   Au = double(Au);
   UU = double(U);
   Daw = Au * inv(UU)

   Daw1 = Daw(1:2,:);
   Daw2 = Daw(3,:);

   save anti_file Daw;
```

（2）Simulink 主程序：chap5_2sim. mdl



（3）控制器程序：chap5_2ctrl. m

```
% LQG controller with AW
function [sys,x0,str,ts] = s_function(t,x,u,flag)
switch flag,
case 0,
    [sys,x0,str,ts] = mdlInitializeSizes;
case 1,
    sys = mdlDerivatives(t,x,u);
case 3,
    sys = mdlOutputs(t,x,u);
```

```
case {1,2, 4, 9 }
    sys = [];
otherwise
    error(['Unhandled flag = ',num2str(flag)]);
end
function [sys,x0,str,ts] = mdlInitializeSizes
sizes = simsizes;
sizes.NumContStates   = 3;
sizes.NumDiscStates   = 0;
sizes.NumOutputs      = 2;
sizes.NumInputs       = 5;
sizes.DirFeedthrough  = 1;
sizes.NumSampleTimes  = 1;
sys = simsizes(sizes);
x0 = [0 0 0];
str = [];
ts = [0 0];
function sys = mdlDerivatives(t,x,u)
w = 1.5 * square(0.25 * t);
y = u(5);
v1 = [u(1) u(2) u(3)]';

 % From chap5_1LQG.m
Ac = [ - 0.0081 1.0000 0; - 203.8428  - 74.1936 244.9490;
            0                   0                  0];
Bcy = [0.0081;0.5; - 1];

Bcw = [0     0;
     0      0;
     0      1];
sys(1:3) = Ac * x + Bcy * y + Bcw(:,2) * w + v1;
function sys = mdlOutputs(t,x,u)
w = 1.5 * square(0.25 * t);
v2 = u(4);
y = u(5);
 % From chap5_1LQG.m
Cc = [ - 202.3428  - 64.1936 244.9490];
Dcy = 0;
Dcw = [0 0];

xc = [x(1) x(2) x(3)]';
ut = Cc * xc + Dcy * y + Dcw(:,2) * w + v2;

sys(1) = w;
sys(2) = ut;
```

(4）补偿程序：chap5_2cmp.m

```
function [sys,x0,str,ts] = s_function(t,x,u,flag)
switch flag,
case 0,
```

```
    [sys,x0,str,ts] = mdlInitializeSizes;
case 3,
    sys = mdlOutputs(t,x,u);
case {2, 4, 9 }
    sys = [];
otherwise
    error(['Unhandled flag = ',num2str(flag)]);
end
function [sys,x0,str,ts] = mdlInitializeSizes
sizes = simsizes;
sizes.NumContStates   = 0;
sizes.NumDiscStates   = 0;
sizes.NumOutputs      = 4;
sizes.NumInputs       = 1;
sizes.DirFeedthrough  = 1;
sizes.NumSampleTimes  = 0;
sys = simsizes(sizes);
x0 = [];
str = [];
ts = [];
function sys = mdlOutputs(t,x,u)
persistent Daw
if t == 0
    load anti_file; % Daw
end
% Daw = [ 0.0038 -0.0395 -0.0118 0.9990];
q = u(1);
v = Daw * q;
sys(1:4) = v;
```

(5) 被控对象程序：chap5_2plant.m

```
function [sys,x0,str,ts] = s_function(t,x,u,flag)
switch flag,
case 0,
    [sys,x0,str,ts] = mdlInitializeSizes;
case 1,
    sys = mdlDerivatives(t,x,u);
case 3,
    sys = mdlOutputs(t,x,u);
case {2, 4, 9 }
    sys = [];
otherwise
    error(['Unhandled flag = ',num2str(flag)]);
end
function [sys,x0,str,ts] = mdlInitializeSizes
sizes = simsizes;
sizes.NumContStates   = 2;
sizes.NumDiscStates   = 0;
sizes.NumOutputs      = 1;
sizes.NumInputs       = 1;
```

```
sizes.DirFeedthrough  = 0;
sizes.NumSampleTimes  = 1;
sys = simsizes(sizes);
x0 = [0 0];
str = [];
ts = [0 0];
function sys = mdlDerivatives(t,x,u)
ut = u(1);
wn = 0; % 0.001 * randn(1,100);
Ap = [0 1; -1  -10];
Bpu = [0;1];
Cpy = [1 0];
Dpyu = 0;
Bpw0 = Bpu;
sys(1:2) = Ap * x + Bpu * ut + Bpw0 * wn;
function sys = mdlOutputs(t,x,u)
Cpy = [1 0];
Dpyw0 = 1;
vn = 0; % 0.0001 * randn(1,100);
sys = Cpy * x + Dpyw0 * vn;      % th
```

(6) 作图程序：chap5_2plot.m

```
close all;

figure(1);
plot(t,w,'-.r',t,y1,'k','linewidth',2);
xlabel('time(s)');ylabel('x1 response');

figure(2);
plot(t,ut(:,1),'r','linewidth',2);
xlabel('time(s)');ylabel('Control input');
```

参考文献

[1] Zaccarian L，Teel A R. Modern anti-windup synthesis：control augmentation for actuator saturation[M]. Princeton：Princeton University Press，2011.

[2] Grimm G，Hatfield J，Postlethwaite I，et al. Antiwindup for stable linear systems with input saturation：an LMI-based synthesis[J]. IEEE Transactions on Automatic Control ，2003,48(9)：1509-1525.

[3] Ailon A. Simple tracking controllers for autonomous VTOL aircraft with bounded inputs[J]. IEEE Transactions on Automatic Control，2010,55(3):737-743.

[4] Wen C Y，Zhou J，Liu Z T，et al. Robust adaptive control of uncertain nonlinear systems in the presence of input saturation and external disturbance[J]. IEEE Transactions on Automatic Control，2011，56(7)：1672-1678.

[5] Hu T S，Teel A，Zaccarian L. Stability and performance for saturated systems via quadratic and non-quadratic lyapunov functions[J]. IEEE Transactions on Automatic Control，2006,51(3)：1770-1786.